Mathematics 2L — Linear Modelling

Solutions 1

1 Only the following of these products exist:

$$AB = \begin{bmatrix} -9 & -3 \\ -24 & -15 \end{bmatrix}$$

$$BA = \begin{bmatrix} -11 & 9 & 40 \\ 8 & 3 & -10 \\ -4 & -12 & -16 \end{bmatrix}$$

$$AC = \begin{bmatrix} 10 & 5 & 4 \\ 8 & 4 & 13 \end{bmatrix}$$

$$CB = \begin{bmatrix} 6 & -3 \\ 6 & 12 \\ -5 & 3 \end{bmatrix}$$

2

$$A^2 = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = A, \quad A^3 = A(A^2) = AA = A^2 = A,$$

$$B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2, \quad B^3 = B(B^2) = BI_2 = B,$$

$$C^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C^3 = C(C^2) = CO = O.$$

3 Let $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ be such a matrix. Then X commutes with A if and only if

$$AX = XA,$$

ie if and only if

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}.$$

Thus we have to solve the matrix equation

$$\begin{bmatrix} x - z & y - w \\ 2x + 3z & 2y + 3w \end{bmatrix} = \begin{bmatrix} x + 2\,y & -x + 3\,y \\ z + 2w & -z + 3w \end{bmatrix}$$

which is equivalent to

$$AX - XA = \begin{bmatrix} -2y - z & x - 2y - w \\ 2x + 2z - 2w & 2y + z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The two diagonal terms are equivalent and give a single equation $2y + z = 0$. Thus we are reduced to solving the system

$$\begin{cases} 2y + z & = 0, \\ 2x + 2z - 2w & = 0, \\ x - 2y - w & = 0. \end{cases}$$

Using first year methods we obtain the general solution

$$z = -2y, \quad w = x - 2y \quad \text{for } x, y \in \mathbb{R}.$$

So the real matrices commuting with A are those of the form

$$\begin{bmatrix} x & y \\ -2y & x - 2y \end{bmatrix} = x \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \qquad \text{for } x, y \in \mathbb{R}.$$

Note that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$. Significance of this?

4 (i) Setting $A = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, we need to solve

$$A^2 = \begin{bmatrix} x^2 + yz & xy + yw \\ zx + wz & yz + w^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

This gives the system of four equations

$$\begin{cases} x^2 + yz &= -1 \\ (x+w)y &= 0 \\ (x+w)z &= 0 \\ yz + w^2 &= -1. \end{cases}$$

Take the second of these equations first.
If $x + w \neq 0$, then we have $y = z = 0$. But this implies that $x^2 = -1$, which is impossible if x is real. Hence $x + w = 0$, giving $w = -x$ and

$$x^2 + yz = -1,$$

so

$$yz = -(1 + x^2) < 0.$$

Thus $y \neq 0$ and

$$z = \frac{-(1 + x^2)}{y},$$

from which we obtain

$$A = \begin{bmatrix} x & y \\ \dfrac{-(1 + x^2)}{y} & -x \end{bmatrix}$$

where $x, y \in \mathbb{R}$ and $y \neq 0$.

(ii) Take $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$: we have to solve

$$\begin{cases} x^2 + yz & = 0 \\ (x+w)y & = 0 \\ (x+w)z & = 0 \\ yz + w^2 & = 0. \end{cases}$$

As above take the second equation first.
Now $y = 0$ gives $x = 0$, $w = 0$ and z arbitrary as one solution. Otherwise $w = -x$ and the equations reduce to $x^2 + yz = 0$ so the solution in this case is

$$A = \begin{bmatrix} x & y \\ \dfrac{-x^2}{y} & -x \end{bmatrix}$$

where $x, y \in \mathbb{R}$ and $y \neq 0$.

5 (i)

$$
\begin{aligned}
[A|I_3] &= \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{bmatrix} \qquad R_2 \longrightarrow R_2 + \frac{1}{2}R_1 \\
&\sim \begin{bmatrix} 2 & 0 & 0 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix} \qquad R_1 \longrightarrow R_1 + \frac{2}{3}R_2,\ R_3 \longrightarrow R_3 + \frac{2}{3}R_2 \\
&\sim \begin{bmatrix} 2 & 0 & 0 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix} \qquad R_1 \longrightarrow R_1 + \frac{1}{2}R_3,\ R_2 \longrightarrow R_2 + \frac{3}{4}R_3 \\
&\sim \begin{bmatrix} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix} \qquad R_1 \longrightarrow \frac{1}{2}R_1,\ R_2 \longrightarrow R_2 + \frac{2}{3}R_2,\ R_3 \longrightarrow \frac{3}{4}R_3
\end{aligned}
$$

giving

$$
A^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}.
$$

5 (ii) No inverse.

5 (iii)

$$A^{-1} = \begin{bmatrix} 6 & 0 & -1 \\ 0 & \frac{5}{6} & -\frac{1}{2} \\ -1 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

6 (i)

$$[A|\mathbf{b}] = \begin{bmatrix}2 & -3 & 0 & 3\\ 4 & -5 & 1 & 7\\ 2 & -1 & -3 & 5\end{bmatrix} \sim \begin{array}{c} \\ 2\\ 1\end{array} \begin{bmatrix}2 & -3 & 0 & 3\\ 0 & 1 & 1 & 1\\ 0 & 2 & -3 & 2\end{bmatrix} \sim \begin{array}{c} \\ \\ 2\end{array} \begin{bmatrix}2 & -3 & 0 & 3\\ 0 & 1 & 1 & 1\\ 0 & 0 & -5 & 0\end{bmatrix}.$$

Back substitution now gives $z = 0$, $y = 1$, $x = 3$.

6 (ii)

$$[A|\mathbf{b}] = \begin{bmatrix}1 & 1 & 1 & 2\\ 1 & 3 & 3 & 0\\ 1 & 3 & 5 & 2\end{bmatrix} \sim \begin{array}{c} \\ 1\\ 1\end{array} \begin{bmatrix}1 & 1 & 1 & 2\\ 0 & 2 & 2 & -2\\ 0 & 2 & 4 & 0\end{bmatrix} \sim \begin{array}{c} \\ \\ 1\end{array} \begin{bmatrix}1 & 1 & 1 & 2\\ 0 & 2 & 2 & -2\\ 0 & 0 & 2 & 2\end{bmatrix}.$$

Back substitution now gives $w = 1$, $v = -2$, $u = 3$.

6 (iii)

$$[A|\mathbf{b}] = \begin{bmatrix}1 & 2 & 2 & 0\\ 3 & -3 & 2 & 5\\ 2 & 4 & -1 & -5\end{bmatrix} \sim \begin{array}{c} \\ 3\\ 2\end{array} \begin{bmatrix}1 & 2 & 2 & 0\\ 0 & -9 & -4 & 5\\ 0 & 0 & -5 & -5\end{bmatrix}.$$

Back substitution now gives $c = 1$, $b = -1$, $a = 0$.

7 Forming the augmented matrix and performing *eros* gives

$$\begin{aligned}
\begin{bmatrix}a & 1 & 2\\ 4 & a & 2\end{bmatrix} &\sim \begin{bmatrix}4 & a & 2\\ a & 1 & 1\end{bmatrix} \qquad R_1 \longleftrightarrow R_2\\
&\sim \begin{bmatrix}4 & a & 2\\ 0 & 1-\frac{a^2}{4} & 1-\frac{a}{2}\end{bmatrix} \qquad R_2 \longrightarrow R_2 - \frac{a}{4}R_1\\
&= \begin{bmatrix}4 & a & 2\\ 0 & (1-\frac{a}{2})(1+\frac{a}{2}) & 1-\frac{a}{2}\end{bmatrix}
\end{aligned}$$

from which one sees that there is no solution precisely when $a = -2$.

8 By Gaussian elimination, or otherwise, we have

$$A = \begin{bmatrix} 1 & 2 & 10 \\ 2 & -8 & -1 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & \frac{-5}{12} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 10 \\ 0 & -12 & -21 \\ 0 & 0 & \frac{13}{4} \end{bmatrix}$$

so

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & \frac{-5}{12} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 & 10 \\ 0 & -12 & -21 \\ 0 & 0 & \frac{13}{4} \end{bmatrix}.$$

Setting

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = U \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

the equation to be solved becomes

$$L \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 15 \\ 24 \\ -6 \end{bmatrix}$$

which gives the system

$$\begin{array}{llll} u & & & = 15 \\ 2u & +v & & = 24 \\ -u & -\frac{5}{12}v & +w & = -6 \end{array}$$

Forward substitution gives $u = 15$, $v = -6$, $w = 13/2$. Thus x, y, z satisfy the system

$$\begin{array}{llll} x & +2y & +10z & = 15 \\ & -12y & -21z & = -6 \\ & & \frac{13}{4}z & = \frac{13}{2} \end{array}$$

from which back substitution gives $x = 1$, $y = -3$, $z = 2$.

9 This time we have

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Setting

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = U \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

the equation to be solved becomes

$$L \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

which gives the system

$$\begin{array}{rrrl} u & & & = 1 \\ -u & +v & & = 2 \\ & 2v & +w & = 4 \end{array}$$

with solution $u = 1$, $v = 3$, $w = -2$. Thus x, y, z satisfy the system

$$\begin{array}{rrrl} 2x & +y & -z & = 1 \\ & 3y & +z & = 3 \\ & & 2z & = -2 \end{array}$$

which has solution $x = -2/3$, $y = 4/3$, $z = -1$.

10 *Repeat* **8** *with partial pivoting —*

$$\begin{aligned} A &= \begin{bmatrix}1&2&10\\2&-8&-1\\-1&3&2\end{bmatrix} R_1\leftrightarrow R_2 \\ &\sim \begin{bmatrix}2&-8&-1\\1&2&10\\-1&3&2\end{bmatrix}\\ &\sim \begin{array}{r} \\ \frac{1}{2}\\ -\frac{1}{2}\end{array}\begin{bmatrix}2&-8&-1\\0&6&\frac{21}{2}\\0&-1&\frac{3}{2}\end{bmatrix}\\ &\sim \begin{array}{r} \\ \\ -\frac{1}{6}\end{array}\begin{bmatrix}2&-8&-1\\0&6&\frac{21}{2}\\0&0&\frac{13}{4}\end{bmatrix}. \end{aligned}$$

We then have $PA = LU$ where

$$L = \begin{bmatrix}1&0&0\\\frac{1}{2}&1&0\\-\frac{1}{2}&-\frac{1}{6}&1\end{bmatrix}, \quad U = \begin{bmatrix}2&-8&-1\\0&6&\frac{21}{2}\\0&0&\frac{13}{4}\end{bmatrix}, \quad P = \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$$

Note that the row interchange comes before we do anything so it does not change L.

We now have $PA\mathbf{x} = LU\mathbf{x} = P\mathbf{b}$. Setting

$$\begin{bmatrix}u\\v\\w\end{bmatrix} = U\begin{bmatrix}x\\y\\z\end{bmatrix},$$

the equation to be solved becomes

$$L\begin{bmatrix}u\\v\\w\end{bmatrix} = P\mathbf{b} = \begin{bmatrix}24\\15\\-6\end{bmatrix}$$

which gives the system

$$\begin{array}{llll} u & & & = 24\\ \frac{u}{2} & +v & & = 15\\ -\frac{u}{2} & -\frac{v}{6} & +w & = -6 \end{array}$$

with solution $u = 24$, $v = 3$, $w = \frac{13}{2}$. Thus x, y, z satisfy the system

$$\begin{array}{rrrl} 2x & -8y & -z & = 24 \\ & 6y & +\frac{21}{2}z & = 3 \\ & & \frac{13}{4}z & = \frac{13}{2} \end{array}$$

which has solution $x = 1$, $y = -3$, $z = 2$.

Repeat **9** *with partial pivoting —*

$$\begin{aligned} A &= \begin{bmatrix} 2 & 1 & -1 \\ -2 & 2 & 2 \\ 0 & 6 & 4 \end{bmatrix} \\ &\sim \begin{array}{r} \\ -1 \\ 0 \end{array} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 6 & 4 \end{bmatrix} R_2 \longleftrightarrow R_3 \\ &\sim \begin{bmatrix} 2 & 1 & -1 \\ 0 & 6 & 4 \\ 0 & 3 & 1 \end{bmatrix} \\ &\sim \begin{array}{r} \\ \\ \frac{1}{2} \end{array} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 6 & 4 \\ 0 & 0 & -1 \end{bmatrix}. \end{aligned}$$

We then have $PA = LU$ where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & \frac{1}{2} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 6 & 4 \\ 0 & 0 & -1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Note that the row interchange comes after the first column of L so it interchanges the second and third rows of column 1 in L.
We now have $PA\mathbf{x} = LU\mathbf{x} = P\mathbf{b}$. Setting

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = U \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

the equation to be solved becomes

$$L\begin{bmatrix}u\\v\\w\end{bmatrix} = P\mathbf{b} = \begin{bmatrix}1\\4\\2\end{bmatrix}$$

which gives the system

$$\begin{array}{rrrl} u & & & = 1 \\ & v & & = 4 \\ -u & +\frac{1}{2}v & +w & = 2 \end{array}$$

with solution $u = 1$, $v = 4$, $w = 1$. Thus x, y, z satisfy the system

$$\begin{array}{rrrl} 2x & +y & -z & = 1 \\ & 6y & +4z & = 4 \\ & & -z & = 1 \end{array}$$

which has solution $x = -\frac{2}{3}$, $y = \frac{4}{3}$, $z = -1$.