Mathematics 2L — Linear Modelling

Problems 5

1. Consider the difference equations

$$y_{k+1} = 0.8y_k + 0.3z_k,$$
$$z_{k+1} = 0.2y_k + 0.7z_k,$$

with initial conditions $y_0 = 0$ and $z_0 = 5$. By diagonalising an appropriate Markov matrix, find formulæ for y_k and z_k and also their limiting values as $k \to \infty$.

2. "Cheap'n Nasty"', a self drive vehicle hire firm, has three depots, located in Glasgow, Manchester and Birmingham. Each month, on average, half those starting off in Glasgow remain there, 40% end up in Birmingham and 10% in Manchester. Of those who start in Manchester, 40% end up there, 50% in Birmingham and 10% in Glasgow. Similarly, 20% of those who start in Birmingham stay there, 40% go to Glasgow and 40% to Manchester.

(i) Formulate this situation as a Markov process with matrix A.

(ii) Find the eigenvalues and eigenvectors of A.

(iii) Find the limiting distribution of the vehicles.

(iv) Find the distribution at month k starting with N_0 in each of the three locations.

3. There is an epidemic with no cure for those infected and in which every month half of those who are well become ill, and a quarter of those who are ill die. Find the limiting steady state solution of this Markov process.

4. Three dairies, McGregors, Lintons and Weirs, supply all of the milk consumed by a particular town. Over a period of a year it is found that McGregors retain 80% of their customers while 10% move to Lintons and 10% to Weirs. Lintons retain 70%, 20% move to McGregors and 10% move to Weirs. Weirs retain only 60%, 30% move to Lintons and 10% move to McGregors. We can assume that the customer base is static.

If the total number of customers for each dairy is currently McGregors: 5,200, Lintons: 6,320, Weirs: 8,340, what will the figures be in three years time? Assuming this trend continues, find the long term outcome.

5. Determine the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}.$$

Hence find the general solution of the system of equations

$$\frac{d\mathbf{q}}{dt} = A\mathbf{q}.$$

6. Differential equations describing a fox and rabbit ecosystem are

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u},$$

with

$$A = \begin{bmatrix} 1 & 8 \\ -1 & 7 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} r \\ f \end{bmatrix},$$

where r and f are the sizes of the two populations at time t.

(i) Solve the system of equations for arbitrary initial conditions by first finding the eigenvalues of A.

(ii) If initially $r = 50$ and $f = 300$, determine the populations at time t.

(iii) What is the long term ratio of rabbits to foxes?