Mathematics 2L — Linear Modelling

Problems 4

1. Evaluate the following determinants:

$$\begin{vmatrix} 3 & 1 & -2 \\ -1 & 4 & -5 \\ 4 & 0 & 1 \end{vmatrix}, \quad \begin{vmatrix} -2 & 1 & 2 \\ 4 & 0 & 1 \\ -3 & 5 & 2 \end{vmatrix}, \quad \begin{vmatrix} x & -x & 4x \\ y & 2y & 0 \\ -z & 3z & 2z \end{vmatrix}.$$

2. Evaluate

$$\begin{vmatrix} 2 & 1 & 0 & -3 \\ 1 & -2 & 4 & 5 \\ 3 & 0 & 1 & 4 \\ -2 & 0 & 3 & 1 \end{vmatrix},$$

by (i) expanding along row 3, (ii) expanding down column 2, (iii) by using *ero*s and/or *eco*s to simplify.

3. In this question all the matrices A, B, C, D and Q are square of the same size, $n \times n$.
Are the following assertions true or false? (Give a supporting reason if true and a counterexample if false.)

(i) If A is invertible and B is singular then $A + B$ is invertible.

(ii) If A is invertible and B is singular then AB is singular.

(iii) Suppose $CD = -DC$. Then

- (a) $\det(C)\det(D) = -\det(D)\det(C)$.
- (b) One, at least, of C or D must be singular.

(iv) If Q is an orthogonal matrix (*ie* $Q^TQ = I$) then $\det(Q) = \pm 1$.

4. Use elementary row or column operations to factorise the following determinant and hence solve the equation

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & x^3 & x \\ x & x & x^3 \end{vmatrix} = 0.$$

5. Find eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} -1 & 2 & -3 \\ 0 & 1 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$

Deduce those for

$$B = \begin{bmatrix} 2 & 2 & -3 \\ 0 & 4 & -1 \\ 1 & -1 & 5 \end{bmatrix}.$$

6. Find the eigenvalues of the matrices

$$A = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 4 & 0 \\ 2 & 5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 15 & 10 & -8 \\ -10 & -5 & 6 \\ 14 & 10 & -7 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & -2 & 3 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{bmatrix}.$$

7. Find the eigenvalues and their corresponding eigenvectors for each of the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 1 & 1 \\ 4 & -4 & -2 \end{bmatrix}.$$

8. Let

$$B = \begin{bmatrix} 2 & 1 & -1 \\ 2 & 6 & -8 \\ 1 & 4 & -6 \end{bmatrix}.$$

(i) Find the eigenvalues of B.

(ii) Find an eigenvector corresponding to the largest eigenvalue.

(iii) Evaluate B^2 and deduce the eigenvalues of the matrix

$$C = \begin{bmatrix} 35 & 12 & -12 \\ 24 & 38 & -6 \\ 12 & 3 & 29 \end{bmatrix}.$$

9. Where possible, diagonalise the following matrices:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

10. Find a matrix A whose eigenvalues are 1 and 4 and whose eigenvectors are $[3, 1]^T$ and $[2, 1]^T$ respectively.

11. Find all of the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, and write down a diagonal matrix D and *two different* diagonalising matrices S, T corresponding to the diagonal matrix D.

12. Which of these matrices *cannot* be diagonalised?

$$\begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}.$$

13. If $A = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}$, diagonalise A and use the result to find A^{100}.