Part 5

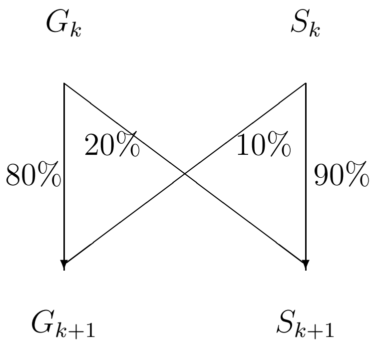
1 Modelling with Eigenvalues

1.1 Markov Processes

Consider the *population-moving scenario*:

Every year 20% of the population of Glasgow leave (to the other parts of Scotland) and in this same period 10% of the people from the rest of Scotland move to Glasgow.

Let us write G_k for the population of Glasgow in year k, and S_k for the population of the rest of Scotland in the same year. Then



$$G_{k+1} = 0.8G_k + 0.1S_k$$

$$S_{k+1} = 0.2G_k + 0.9S_k$$

Let us write

$$\mathbf{u}_k = \begin{bmatrix} G_k \\ S_k \end{bmatrix}.$$

Our scenario is described by the *recurrence relation*

$$\mathbf{u}_{k+1} = A\mathbf{u}_k \quad \text{where} \quad A = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}.$$

It is clear that the evolution of this scenario is

$$\mathbf{u}_n = A^n \mathbf{u}_0.$$

For this A

$$\det(A - \lambda I) = \begin{vmatrix} 0.8-\lambda & 0.1 \\ 0.2 & 0.9-\lambda \end{vmatrix} = \lambda^2 - 1.7\lambda + 0.7 = (\lambda - 1)(\lambda - 0.7)$$

so the eigenvalues are 1 and 0.7.

For $\lambda = 1$

$$\begin{aligned} A - I &= \begin{bmatrix} -0.2 & 0.1 \\ 0.2 & -0.1 \end{bmatrix} \\ &\sim \begin{bmatrix} -0.2 & 0.1 \\ 0 & 0 \end{bmatrix} \\ &\sim \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

so an 1-eigenvector is $[1, 2]^{\mathrm{T}}$.

For $\lambda = 0.7$

$$\begin{aligned} A - 0.7\, I &= \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.2 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

so an 0.7-eigenvector is $[1, -1]^{\mathrm{T}}$.

Thus A is diagonalisable, with $D = \begin{bmatrix} 1 & 0 \\ 0 & 0.7 \end{bmatrix}$ and $S = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$. Also

$$S^{-1} = \frac{1}{-1-2} \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix},$$

whence

$$\begin{aligned} A^n &= SD^nS^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.7 \end{bmatrix}^n \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & 0.7^n \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 1^n + 2(0.7)^n & 1^n - 0.7^n \\ 2(1^n - 0.7^n) & 2(1^n) + 0.7^n \end{bmatrix}. \end{aligned}$$

The solution is $\mathbf{u}_n = A^n \mathbf{u}_0$, *ie*

$$\begin{aligned} \begin{bmatrix} G_n \\ S_n \end{bmatrix} &= \frac{1}{3} \begin{bmatrix} [1^n + 2(0.7)^n]G_0 + [1^n - 0.7^n]S_0 \\ 2(1^n - 0.7^n)G_0 + [2(1^n) + 0.7^n]S_0 \end{bmatrix} \\ &= \frac{1^n}{3}(G_0 + S_0) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{0.7^n}{3}(2G_0 - S_0) \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \end{aligned}$$

Since $0.7^n \to 0$ as $n \to \infty$ we see that $\mathbf{u}_n \to \dfrac{G_0 + S_0}{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as $n \to \infty$: *ie*

$$G_n \to \frac{1}{3}(G_0 + S_0), \qquad S_n \to \frac{2}{3}(G_0 + S_0).$$

It is natural to write

$$\begin{aligned} A^n &= \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & 0.7^n \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \\ &\longrightarrow \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \end{aligned}$$

which again shows that

$$G_n \to \frac{1}{3}(G_0 + S_0), \qquad S_n \to \frac{2}{3}(G_0 + S_0).$$

These limits depend only on the *sum* $G_0 + S_0$. They are in the same proportion as the entries in any 1-eigenvector.

Processes like this one, where each state is determined by its immediate predecessor, and where numbers are positive and their total is conserved, are called *Markov processes*. The matrix describing such a process, sometimes called the *transition matrix*, is a *Markov matrix*: a square matrix all of whose entries are ≥ 0 and all its column sums equal to 1.

Column & row sums

Note that if A is an $m \times n$ matrix

$$A = [\,\mathbf{c}_1 : \mathbf{c}_2 : \cdots : \mathbf{c}_n] = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_m \end{bmatrix}$$

then

$$[1 : 1 : \cdots : 1]A = \left[\sum \mathbf{c}_1 : \sum \mathbf{c}_2 : \ldots : \sum \mathbf{c}_n\right]$$

while

$$A \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \sum \mathbf{r}_1 \\ \sum \mathbf{r}_2 \\ \vdots \\ \sum \mathbf{r}_m \end{bmatrix}$$

In particular, if A is an $n \times n$ Markov matrix then

$$A^{\mathrm{T}}[1, 1, \ldots, 1]^{\mathrm{T}} = [1, 1, \ldots, 1]^{\mathrm{T}}\,.$$

So 1 is automatically an eigenvalue of A^{T}, and therefore of A too.

Trends

We have seen that the evolution of the population-moving scenario shows a *trend* to a fixed final proportion. This is not coincidental.

1.2 Positive matrices & Markov processes

Say that a matrix $A = [a_{ij}]$ is *nonnegative* if each entry $a_{ij} \geq 0$, and *positive* if each entry is > 0. This is not the same as being *positive (semi)definite.*

Suppose that A is a positive square matrix and that

$$\mu = \max\{|\,\lambda\,| : \lambda \textit{ an eigenvalue of } A\}$$

Then μ is an eigenvalue of A and there is a positive eigenvector $\mathbf{p}$ corresponding to μ: moreover $\mathbf{p}$ is essentially unique [every other μ-eigenvector is a scalar multiple of $\mathbf{p}$ — we say that μ is an eigenvalue of *multiplicity one*].

Also, every other eigenvalue has modulus $< \mu$.

Further

$$\min_i \sum_j a_{ij} \leq \mu \leq \max_i \sum_j a_{ij}$$

ie μ lies between the least and largest *row sums*: similarly for *column sums.*

Also, for any $\mathbf{x}$, we have

$$(\mu^{-1} A)^k \,\mathbf{x} \longrightarrow_{k\to\infty} \text{const } \mathbf{p}$$

for some constant.

The same results hold for a nonnegative matrix A, so long as some power A^k is positive.

Markov matrices

Suppose that M is a Markov matrix. Then $\mu = 1$ is an eigenvalue and any 1-eigenvector can be scaled to be ≥ 0.

Every other eigenvalue λ satisfies $|\,\lambda\,| \leq 1$.

If, further, $M^k > 0$ for some k, then all other eigenvalues satisfy $|\,\lambda\,| < 1$: moreover, M has essentially only one eigenvector, say $\mathbf{p}$, corresponding to the eigenvalue 1 and then, for *any* (nonzero) initial vector $\mathbf{u}$,

$$M^n\mathbf{u} \to \text{const}\ \mathbf{p}$$

as $n \to \infty$.

BUT there is no long-term trend for the Markov matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Eigenvalues of 2×2 *Markov matrices*

If A is a 2×2 Markov matrix then:

1 1 is an eigenvalue, and

2 the sum of the (two) eigenvalues is equal to $\text{tr}\,(A)$ *ie* to $a_{11} + a_{22}$.

So the *other eigenvalue* is $a_{11} + a_{22} - 1$.

Revisiting the *population-moving scenario*:

$$\text{tr}\,A = 0.8 + 0.9 = 1.7$$

from which it is evident that since 1 is an eigenvalue the other eigenvalue must be $1.7 - 1$ *ie* 0.7.

E A transport manager has a large pool of cars. 60% are Ws, 30% are Ms, and 10% are Ds. Each month some cars are sold, others bought, the pool being kept constant in size. In the last month

$$\begin{array}{lllllll} W & \rightarrow & 90\% \text{ remain } W & \& & 10\% & \rightarrow M & \\ M & \rightarrow & 80\% \text{ remain } M & \& & 5\% & \rightarrow D & \& \; 15\% \rightarrow W \\ D & \rightarrow & 70\% \text{ remain } D & \& & 20\% & \rightarrow W & \& \; 10\% \rightarrow M \end{array}$$

What would be the long-term outcome of this trend, if it were to continue?

S In our usual style, let W_k be the number of cars of type W in month k, *etc*, and let $\mathbf{u}_k = \begin{bmatrix} W_k \\ M_k \\ D_k \end{bmatrix}$.

Then

$$\mathbf{u}_{k+1} = A\mathbf{u}_k,$$

where

$$A = \begin{bmatrix} 0.9 & 0.15 & 0.2 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.05 & 0.7 \end{bmatrix}.$$

Although A has a zero entry its square does not — *check*.

Since this A is a Markov matrix it has 1 as an eigenvalue. To find them all:

$$\begin{aligned} |A - \lambda I_3| &= \begin{vmatrix} 0.9-\lambda & 0.15 & 0.2 \\ 0.1 & 0.8-\lambda & 0.1 \\ 0 & 0.05 & 0.7-\lambda \end{vmatrix} \\ &= \begin{vmatrix} 1-\lambda & 1-\lambda & 1-\lambda \\ 0.1 & 0.8-\lambda & 0.1 \\ 0 & 0.05 & 0.7-\lambda \end{vmatrix} \qquad \begin{array}{l} R_1 \rightarrow R_1 + R_2 + R_3 \\ \textit{this always produces} \\ \textit{a factor } 1-\lambda \end{array} \\ &= (1-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 0.1 & 0.8-\lambda & 0.1 \\ 0 & 0.05 & 0.7-\lambda \end{vmatrix} \\ &= (1-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 0.1 & 0.7-\lambda & 0 \\ 0 & 0.05 & 0.7-\lambda \end{vmatrix} \qquad \begin{array}{ll} C_2 \rightarrow C_2 - C_1 & \textit{to produce } 0\text{s} \\ C_3 \rightarrow C_3 - C_1 & \textit{in top row} \end{array} \\ &= (1-\lambda)(0.7-\lambda)^2. \end{aligned}$$

As above, the *long-term trend* is that for any initial distribution $\mathbf{u}_0$

$$A^n \mathbf{u}_0 \to \text{const } \mathbf{p},$$

where $\mathbf{p}$ is a [positive] 1-eigenvector. Now

$$\begin{aligned}
A - I_3 &= \begin{bmatrix} -0.1 & 0.15 & 0.2 \\ 0.1 & -0.2 & 0.1 \\ 0 & 0.05 & -0.3 \end{bmatrix} \\
&\sim \begin{bmatrix} -0.1 & 0.15 & 0.2 \\ 0 & -0.05 & 0.3 \\ 0 & 0.05 & -0.3 \end{bmatrix} \\
&\sim \begin{bmatrix} -0.1 & 0.15 & 0.2 \\ 0 & -0.05 & 0.3 \\ 0 & 0 & 0 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & -11 \\ 0 & 1 & -6 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

so $[11, 6, 1]^{\mathrm{T}}$ is a 1-eigenvector.

This means that in the long term the pool will be in the proportions

$$11W : 6M : 1D$$

irrespective of the initial proportions.

1.3 Digression on the e$^{\text{xponential}}$ of a matrix

The exponential of a number

For any complex number z define

$$\exp(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!}$$

It can be shown that this series converges to a finite limit for any z.

For given z and w we have

$$\begin{aligned}
\exp(z)\exp(w) &= \left\{1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right\}\left\{1 + \frac{w}{1!} + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots\right\} \\
&= 1 + \frac{z+w}{1!} + \frac{z^2 + 2zw + w^2}{2!} + \cdots \\
&\qquad + \frac{z^n + \binom{n}{1} z^{n-1} w + \binom{n}{2} z^{n-2} w^2 + \cdots + w^n}{n!} + \cdots \\
&= 1 + \frac{z+w}{1!} + \frac{(z+w)^2}{2!} + \frac{(z+w)^3}{3!} + \cdots \\
&= \exp(z+w)
\end{aligned}$$

Define

$$e = \exp(1)$$

Then

$$e^n = \exp(n)$$

for any integer n ... Also

$$e^z e^{-z} = e^0 = 1 \qquad \textit{for any } z$$

And

$$\frac{e^{z+h} - e^z}{h} = e^z \, \frac{e^h - 1}{h} = e^z \left\{1 + \frac{h}{2!} + \frac{h^2}{3!} + \cdots\right\} \longrightarrow_{h \to 0} \; e^z$$

ie

$$\frac{d}{dz} e^z = e^z$$

The exponential of a square matrix

If A is *any* square matrix we can form the power series

$$I + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots$$

This series converges to a matrix that we shall denote by e^A.

The matrix exponential obeys the *index law for two commuting 'indices'*: if $AB = BA$ we have

$$e^A\, e^B = e^{A+B} = e^B e^A.$$

In particular, for a fixed square A and scalars s, t

$$\begin{aligned} e^{sA}e^{tA} &= e^{(s+t)A} \\ e^{tA}e^{-tA} &= I \\ \frac{d}{dt}\, e^{tA} &= Ae^{tA}\,. \end{aligned}$$

Moreover, if B is square and S [same size] is invertible we find

$$\begin{aligned} e^{SBS^{-1}} &= I + \frac{SBS^{-1}}{1!} + \frac{(SBS^{-1})^2}{2!} + \frac{(SBS^{-1})^3}{3!} + \ldots \\ &= SIS^{-1} + \frac{SBS^{-1}}{1!} + \frac{SB^2S^{-1}}{2!} + \frac{SB^3S^{-1}}{3!} + \ldots \\ &= S\left\{ I + \frac{B}{1!} + \frac{B^2}{2!} + \frac{B^3}{3!} + \ldots \right\} S^{-1} \\ &= Se^BS^{-1}\,. \end{aligned}$$

In particular, when $A = SDS^{-1}$ and $D = \mathrm{diag}\{d_1, \ldots, d_n\}$ we have

$$e^A = e^{SDS^{-1}} = Se^DS^{-1} = S\,\mathrm{diag}\{e^{d_1}, \ldots, e^{d_1}\}\, S^{-1}.$$

end of **digression**

1.4 Solution of $\dot{X} = AX$

Consider a system of differential equations for $\mathbf{u} = [u_1, u_2]^{\mathrm{T}}$: *eg*

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{u} \qquad (*)$$

with initial condition

$$\mathbf{u}(0) = \mathbf{u}_0.$$

Now, for λ and $\mathbf{x}$ fixed,

$$\frac{d}{dt} e^{\lambda t} \mathbf{x} = \lambda e^{\lambda t} \mathbf{x}$$

so $\mathbf{u}(t) = e^{\lambda t}\mathbf{x}$ will provide a solution to $(*)$ so long as

$$\lambda e^{\lambda t} \mathbf{x} = A e^{\lambda t} \mathbf{x}$$

for all t: *ie* if

$$A\mathbf{x} = \lambda \mathbf{x}$$

ie if $(\lambda, \mathbf{x})$ is an eigenpair for A.

Now the eigenvalues of this A are $\lambda_1 = -1$, with $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\lambda_2 = -3$, with $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Since the differential equation $(*)$ is *linear* we see that any linear combination

$$c_1 \, e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \, e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

[where c_1 and c_2 are scalar constants] will also be a solution of $(*)$.

We need to choose c_1, c_2 to fit this solution to the initial condition $\mathbf{u}(0) = \mathbf{u}_0$: *ie*

$$\mathbf{u}_0 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 - c_2 \end{bmatrix}.$$

The matrix $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ can be diagonalised:

$$S^{-1} A S = D$$

where

$$D = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, \qquad S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

The above solution can be rewritten:

$$c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = S\, e^{tD}\, \mathbf{c}$$

where $\mathbf{c} = [c_1, c_2]^{\mathrm{T}}$.

Now we see that to fit the solution to the initial conditions we need $\mathbf{c}$ to satisfy

$$Se^0\mathbf{c} = \mathbf{u}_0$$

ie

$$\mathbf{c} = S^{-1}\mathbf{u}_0.$$

Then

$$\mathbf{u}(t) = Se^{tD}\mathbf{c} = Se^{tD}S^{-1}\mathbf{u}_0 = e^{tA}\mathbf{u}_0.$$

Diagonalise first ?

If we start by diagonalising A we can rephrase the equation $(*)$ as

$$\frac{d}{dt}\mathbf{u} = A\mathbf{u} = SDS^{-1}\mathbf{u}.$$

On changing the variable to

$$\mathbf{v} = S^{-1}\mathbf{u}$$

we find

$$\frac{d}{dt}\mathbf{v} = S^{-1}\frac{d}{dt}\mathbf{u} = S^{-1}A\mathbf{u} = S^{-1}AS\mathbf{v} = D\mathbf{v}$$

(the equations have *uncoupled*) which clearly has solution

$$\mathbf{v}(t) = e^{tD}\mathbf{v}_0$$

with $\mathbf{v}_0 = S^{-1}\mathbf{u}_0$.

So the solution sought is

$$\mathbf{u}(t) = S\mathbf{v}(t) = Se^{tD}\mathbf{v}_0 = Se^{tD}S^{-1}\mathbf{u}_0 = e^{tA}\mathbf{u}_0.$$

1.5 Predator-prey models

E Suppose that the rabbit and fox populations are modelled by the equations

$$\frac{dr}{dt} = 4r - 2f, \qquad \frac{df}{dt} = r + f,$$

and that initially $r_0 = 300$, $f_0 = 20$. What are the populations at time t? What happens in the long term?

SIGNIFICANCE OF SIGNS AND SIZES OF COEFFICIENTS ?

S We can write these equations in matrix form as

$$\frac{d}{dt}\mathbf{u} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \mathbf{u}$$

with

$$\mathbf{u} = \begin{bmatrix} r \\ f \end{bmatrix}, \qquad \mathbf{u}_0 = \begin{bmatrix} r_0 \\ f_0 \end{bmatrix} = \begin{bmatrix} 300 \\ 20 \end{bmatrix}.$$

Eigenvalues of A? Here

$$\begin{aligned} |\, A - \lambda I \,| &= \begin{vmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} \\ &= (4-\lambda)(1-\lambda) + 2 = \lambda^2 - 5\lambda + 6 \\ &= (\lambda - 2)(\lambda - 3). \end{aligned}$$

Since we have n $(= 2)$ *distinct* eigenvalues we can certainly diagonalise A.

Now

$$A - 2I = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$$

so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a 2-eigenvector; while

$$A - 3I = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}$$

so $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a 3-eigenvector.

Put

$$S = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Then

$$S^{-1} = \frac{1}{1-2} \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}.$$

The solution of the system is

$$\begin{aligned} \mathbf{u}(t) &= Se^{tD}S^{-1}\mathbf{u}_0 \\ &= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 300 \\ 20 \end{bmatrix} \\ &= \begin{bmatrix} e^{2t} & 2e^{3t} \\ e^{2t} & e^{3t} \end{bmatrix} \begin{bmatrix} -260 \\ 280 \end{bmatrix} \\ &= \begin{bmatrix} 560e^{3t} - 260e^{2t} \\ 280e^{3t} - 260e^{2t} \end{bmatrix}. \end{aligned}$$

In the long term both r and $f \to \infty$: but the ratio

$$\frac{r}{f} = \frac{560e^{3t} - 260e^{2t}}{280e^{3t} - 260e^{2t}} = \frac{560 - 260e^{-t}}{280 - 260e^{-t}} \to \frac{560}{280} = \frac{2}{1}.$$

Notice that this ratio is $v_1 : v_2$, where $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is any eigenvector corresponding to the larger eigenvalue, *ie* to the *dominating* term of the solution.

Alternatively —

$$\begin{aligned} e^{-3t}\mathbf{u}(t) &= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 300 \\ 20 \end{bmatrix} \\ &\to \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 300 \\ 20 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 300 \\ 20 \end{bmatrix} \\ &= \left(300 - 20\right) \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \end{aligned}$$