Part 3

1 Linear Programming

1.1 Introduction

Consider the following fairly simple problem.

Problem J

A jam factory can make either marmalade or strawberry jam.

Its bottling plant has a maximum capacity of 10 000 *jars per day.*

Each jar requires $\frac{1}{2}$ lb *of fruit.*

*Oranges are three times easier to handle than strawberries: it can handle 6 000*lb *oranges or 2 000*lb *strawberries (or some equivalent combination) per day.*

The factory can sell all it produces. The profit is 2p *on each jar of marmalade,* 3p *on each jar of jam.*

What proportions of jam & marmalade will maximise the profit ?

Just jam? 2 000 *lb* → 4 000 jars → £120 profit.

Just marmalade? 6 000 *lb* → 12 000 jars: but bottling bottleneck restricts to 10 000 jars → £200 profit.

Producing one jar less, *ie* 9 999 jars of marmalade and with it one jar of jam, will boost the profit to £200.01.

Clearly the profit will be maximised by producing some mixture of both.
How to identify a maximising combination?

Mathematical formulation

Let x be the number of jars of marmalade and y be the number of jars of jam produced daily. Then the total profit will be

$$P = 2x + 3y.$$

Constraints

First, $x \geq 0$ & $y \geq 0$.

Next, the capacity of the bottling plant forces

$$x + y \leq 10\,000.$$

Also, the processing constraint [no more than 6 000 *lb* oranges or 2 000 *lb* strawberries] is that

$$\frac{1}{2}x + \frac{3}{2}y \leq 6\,000,$$

ie

$$x + 3y \leq 12\,000.$$

Let us *rescale* and let x, y measure thousands of jars. Then the profit in £s will be

$$P = 10(2x + 3y).$$

So we can *reformulate* —

Problem J

Maximise

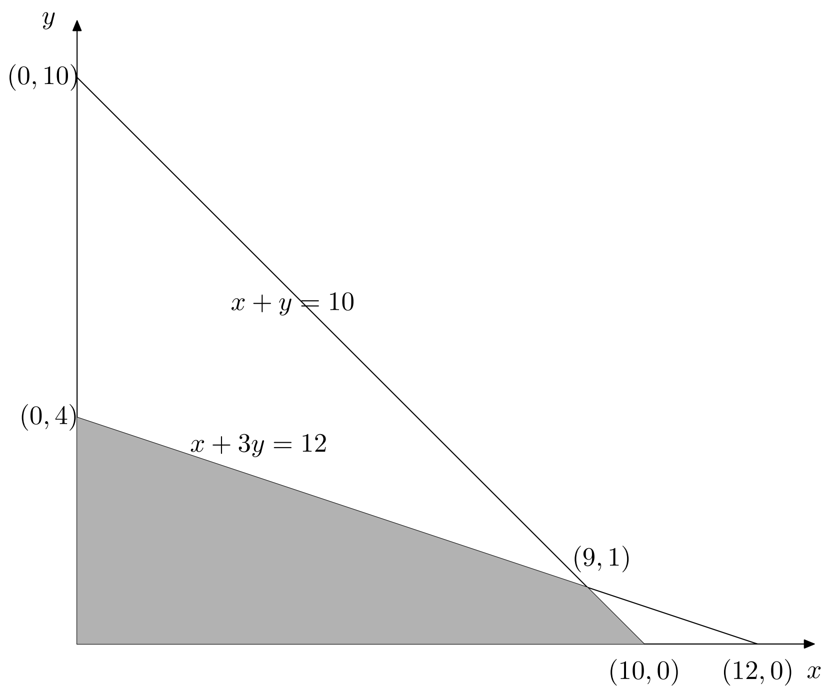
$$P = 2x + 3y$$

subject to

$$\begin{aligned} x &\geq 0, \\ y &\geq 0, \\ x + y &\leq 10, \\ x + 3y &\leq 12. \end{aligned}$$

Graphical solution

The region of the x-y–plane defined by these constraints lies in the first quadrant and is further confined by the two lines $x+y=10$ and $x+3y=12$. It is called the *feasible set*: every point in it is a *feasible point* or *feasible solution.*



A feasible point that maximises the profit is an *optimal point*. In this problem it is the intersection of the two lines $x+y=10$ and $x+3y=12$, *ie* $x=9$, $y=1$.

In terms of the original problem this means that the factory should produce 9 000 jars of marmalade and 1 000 of jam daily. The daily profit will then be $\boxed{\text{£210}}$.

1.2 General Linear Programming Problem

This introductory **Problem J** is one of many that can be framed as follows:

$$\begin{gathered}\substack{\text{max}\\\text{min}}\text{imise} \quad P(x_1, x_2, \ldots, x_n)\\ \text{subject to} \quad A\mathbf{x} \le \mathbf{b}, \ \ \mathbf{x} \ge 0.\end{gathered}$$

The optimal point is a 'corner' [or 'vertex'] of the feasible set. This will always be so if there is only one optimal point. If there are more, then all points in their span will also be optimal.

It is hardly feasible to solve this kind of problem by systematically checking the values of P at the vertices of the feasible set. In realistic problems there may be millions of corners, each costly to find.

1.3 Simplex Method

We start by turning the nontrivial constraints from *inequalities* into *equalities* by introducing *slack variables*, one for each such constraint.

In **Problem J** write x_1 for x and x_2 for y. Introduce the slack variables x_3, x_4 by

$$\begin{aligned} x_3 &= 10 - x_1 - x_2, \\ x_4 &= 12 - x_1 - 3x_2. \end{aligned}$$

The problem can now be restated as

Maximise

$$P(x_1, x_2, x_3, x_4) = 2x_1 + 3x_2$$

subject to

$$[x_1, x_2, x_3, x_4]^{\mathrm{T}} \ge 0$$

and

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \end{bmatrix},$$

which we write as

$$A\mathbf{x} = \mathbf{b}.$$

Effectively, we have replaced the original inequality constraints $A\mathbf{x} \le \mathbf{b}$ by a set of equations $A\mathbf{x} = \mathbf{b}$ with a new A and a new $\mathbf{x}$.

Corners of the feasible set

Suppose we start with a problem having n variables $x_1, x_2, \ldots x_n$ and m inequality constraints. Once we have introduced the m slack variables $x_{n+1}, \ldots x_{m+n}$ we have m equations $A\mathbf{x} = \mathbf{b}$, where A is $m \times (m+n)$, in the $m+n$ unknowns $x_1, x_2, \ldots x_n, x_{n+1}, \ldots x_{m+n}$.

Each corner of the feasible set is a point where n of the $m+n$ components of $\mathbf{x}$ are zero.

In $A\mathbf{x} = \mathbf{b}$ there are m *basic* variables and n *free* variables. If we set the n free variables to be zero we get a basic solution. This is a genuine corner of the feasible set if the other variables are ≥ 0.

If in **Problem J** we put $x_3 = x_4 = 0$ we find $x_1 = 9$ and $x_2 = 1$. This is indeed a corner of the feasible set [and even the optimal point].

How might we determine such a point algebraically?

Let us arrange the description as follows: first the function to be optimised, then the constraints.

$\boxed{}$ indicates the pivot to be used: can be marked only after this has been decided.

$$\begin{array}{cccccccl} P & -2x_1 & -3x_2 & & & = & 0 & (P) \\ & \boxed{x_1} & +x_2 & +x_3 & & = & 10 & (1) \\ & x_1 & +3x_2 & & +x_4 & = & 12 & (2) \end{array}$$

Here x_1, x_2 are free, x_3, x_4 are basic.

Use equation (1) to eliminate x_1 from the other two equations. This gives

$$\begin{array}{cccccccl} P & & -x_2 & +2x_3 & & = & 20 & (P') = (P) + 2(1) \\ & x_1 & +x_2 & +x_3 & & = & 10 & (1) \\ & & \boxed{2x_2} & -x_3 & +x_4 & = & 2 & (2') = (2) - (1) \end{array}$$

and now x_1, x_4 are basic while x_2, x_3 are free. We say that x_1 has *entered* the basis and that x_2 has *left* the basis.

Next eliminate x_2 from the other two equations by using equation (2′):

$$\begin{array}{ccccccccl} 2P & & & +3x_3 & +x_4 & = & 42 & & (P'') = 2(P') + (2') \\ & 2x_1 & & +3x_3 & -x_4 & = & 18 & & (1') = 2(1) - (2') \\ & & 2x_2 & -x_3 & +x_4 & = & 2 & & (2') \end{array}$$

Now x_1, x_2 are basic while x_3, x_4 are free: x_2 has *entered* the basis and x_3 has *left* the basis.

We can now rearrange the [latest form of the] P-equation as

$$P = 21 - \frac{3}{2}x_3 - \frac{1}{2}x_4.$$

Since the feasible set is characterised by $\mathbf{x} \geq 0$ we see that P attains its maximum where $x_3 = x_4 = 0$.

E Minimise

$$C = 16 + 7x_1 - x_2 - 3x_3$$

subject to the constraints

$$\begin{array}{cccccc} x_1 & +6x_2 & & +2x_3 & \leq & 8 \\ x_1 & & & +3x_3 & \leq & 9 \\ & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

We introduce slack variables x_4 and x_5 and rewrite the original constraint inequalities as equations:

$$\begin{array}{cccccccc} x_1 & +6x_2 & +2x_3 & +x_4 & & = & 8 \\ x_1 & & +3x_3 & & +x_5 & = & 9 \end{array}$$

As in our treatment of the previous problem we display the criteria with the C-equation first, followed by the constraint equations.

$$\begin{array}{cccccccccl} C & -7x_1 & +x_2 & +3x_3 & & & = & 16 & (C) \\ & x_1 & +6x_2 & +2x_3 & +x_4 & & = & 8 & (1) \\ & x_1 & & \boxed{+3x_3} & & +x_5 & = & 9 & (2) \end{array}$$

We seek to *minimise* C so we need to express C in the form $c + \sum \lambda_j x_j$ with the $\lambda_j \geq 0$: equivalently, the coefficients of the x variables in the C-equation must be *negative*.

Notice that x_4, x_5 are basic while x_1, x_2, x_3 are free. The choice $x_1 = x_2 = x_3 = 0$ gives $x_4 = 8$ and $x_5 = 9$. These are both positive so we have a point of the feasible set: and there $C = 16$.

We start by choosing x_3 to enter the basis. Which variable should leave?

If we choose x_4 to leave then the new set of free variables will be x_1, x_2, x_4. Putting $x_1 = x_2 = x_4 = 0$ leads to $2x_3 = 8$ and then to $x_5 = 9 - 3x_3 = -3$, not a feasible point.

On the other hand, if we choose x_5 to leave then the new set of free variables will be x_1, x_2, x_5. Putting $x_1 = x_2 = x_5 = 0$ leads to $x_3 = 3$ and then to $x_4 = 2$, a feasible corner. So we choose x_5 to leave.

After using equation (2) to eliminate x_3 from the other two equations we find

$$\begin{array}{llllllll} C & -8x_1 & +x_2 & & & -x_5 & = & 7 & (C') = (C) - (2) \\ & x_1 & \boxed{+18x_2} & & +3x_4 & -2x_5 & = & 6 & (1') = 3(1) - 2(2) \\ & x_1 & & +3x_3 & & +x_5 & = & 9 & (2) \end{array}$$

and $C = 7$ at the corner $x_1 = x_2 = x_5 = 0, x_3 = 3, x_4 = 2$: an improvement.

Now the only variable with a positive coefficient in C is x_2. It must enter. The only constraint equation featuring x_2 is $(1')$: so x_4 leaves. We find

$$\begin{array}{lllllllll} 18C & -145x_1 & & & -3x_4 & -16x_5 & = & 120 & (C'') = 18(C') - (1') \\ & x_1 & +18x_2 & & +3x_4 & -2x_5 & = & 6 & (1') \\ & x_1 & & +3x_3 & & +x_5 & = & 9 & (2) \end{array}$$

This shows that the minimum for C is $\frac{120}{18} = \dfrac{20}{3}$, attained when $x_1 = x_4 = x_5 = 0$, $x_2 = \frac{1}{3}, x_3 = 3$.

Notice that once we have seen that all the signs of the x_k in the C-equation are negative we do not need to compute any more row operations.

Row ratios

Let us look back to when x_3 was to enter the basis, and we had to decide which of x_4, x_5 was to leave.

Each constraint equation featuring the entering variable [here x_3] defines a *row ratio*

$$\left| \frac{\text{right hand side}}{\text{coefficient of the entering variable}} \right| .$$

Here they are $\frac{8}{2} = 4$ and $\frac{9}{3} = 3$. The variable with the smallest row ratio [in modulus] is the one to leave: here it is x_5.

Simplex Method or **Algorithm**

We can now describe the steps of the *Simplex Method* or *Algorithm*.

(i) Introduce slack variables.

(ii) Write down the 'tableau' of function and constraints.

The term 'tableau' is used here in a slightly different way from in the textbooks.

(iii) Decide on an entering variable — usually the one with the largest *adverse* coefficient in the C-equation

(iv) Decide on a leaving variable — the one with the lowest row ratio [if there is a choice]

(v) If the elimination gives an equation for the function to be optimised with the coefficients of all the nonbasic variables of the correct sign, and if the choice of these nonbasic variables to be zero corresponds to a feasible point, then we are done,

so **STOP**:

ELSE return to stage **(iii)**.

E Find the maximum of $V = x_1 + x_2 + x_3$ where $x_1, x_2, x_3 \geq 0$, subject to the constraints $x_1 + 5x_3 \leq 1$, $4x_1 + 7x_2 + 2x_3 \leq 1$.

S Introducing slack variables

$$\begin{aligned} x_4 &= 1 - x_1 - 5x_3 \\ x_5 &= 1 - 4x_1 - 7x_2 - 2x_3 \end{aligned}$$

we obtain the system

$$\begin{array}{cccccccll} V & -x_1 & -x_2 & -x_3 & & & = & 0 & (V) \\ & x_1 & & +5x_3 & +x_4 & & = & 1 & (1) \\ & \boxed{4x_1} & +7x_2 & +2x_3 & & +x_5 & = & 1 & (2) \end{array}$$

Let us start by choosing x_1 as the entering variable [**(iii)** gives us no firm guidance here].

The *row ratios* for x_1 are $\frac{1}{1}$ and $\frac{1}{4}$ so we choose x_5 to leave. We eliminate x_1 from all but (2). This gives

$$\begin{array}{cccccccll} 4V & & +3x_2 & -2x_3 & & +x_5 & = & 1 & (V') = 4(V) + (2) \\ & & -7x_2 & \boxed{+18x_3} & +4x_4 & -x_5 & = & 3 & (1') = 4(1) - (2) \\ & 4x_1 & +7x_2 & +2x_3 & & +x_5 & = & 1 & (2) \end{array}$$

Now x_1, x_4 are basic. The only negative coefficient in (V') is that of x_3. We choose x_3 to enter. The row ratio for x_3 in $(1')$ is $\frac{3}{18} = \frac{1}{6}$, while in (2) it is $\frac{1}{2}$. So we choose x_4 to leave. Now eliminating x_3 from (V') using $(1')$ gives

$$\begin{array}{cccccll} 36V & +20x_2 & +4x_4 & +8x_5 & = & 12 & (V'') = 9(V') + (1') \\ & & & \vdots & & & \end{array}$$

This shows that V attains its maximum, $\dfrac{1}{3}$, when $x_2 = x_4 = x_5 = 0$, so for $x_1 = \frac{1}{6} = x_3$.

E This example has the feature that not all the entries in A are ≥ 0.

Minimise

$$C = 2x_1 + x_2$$

subject to

$$\begin{array}{rrcr} x_1 & +x_2 & \geq & 8 \\ x_1 & +3x_2 & \geq & 12 \\ x_1 & -x_2 & \geq & 0. \end{array}$$

S We introduce slack variables x_3, x_4, x_5 by

$$\begin{array}{rcrrr} x_3 & = & x_1 & +x_2 & -8 \\ x_4 & = & x_1 & +3x_2 & -12 \\ x_5 & = & x_1 & -x_2 & \end{array}$$

Our 'tableau' is

$$\begin{array}{rrrrrrcrr} C & -2x_1 & -x_2 & & & & = & 0 & (C) \\ & x_1 & +x_2 & -x_3 & & & = & 8 & (1) \\ & \boxed{x_1} & +3x_2 & & -x_4 & & = & 12 & (2) \\ & x_1 & -x_2 & & & -x_5 & = & 0. & (3) \end{array}$$

At this first stage the basic variables are the slack variables.

We choose x_1 to enter. If we follow the row-ratio criterion we must take x_5 to leave. Unfortunately this corresponds to the point $x_2 = x_5 = 0$ at which $x_1 = 0$ and the x_3, x_4 are *negative.*

The other two row ratios are 8 and 12: so we look to see whether we can choose x_3 to leave. But this corresponds to the point where $x_4 = -4$ so we ignore this possibility too.

So we take x_4 to leave. This corresponds to the point where $x_1 = 12$, $x_2 = x_4 = 0, x_3 = 4, x_5 = 12$, which is feasible. Eliminating leads to

$$\begin{array}{rrrrrcrl} C & & +5x_2 & & -2x_4 & & = & 24 & (C') = (C) + 2(2) \\ & & \boxed{-2x_2} & -x_3 & +x_4 & & = & -4 & (1') = (1) - (2) \\ & x_1 & +3x_2 & & -x_4 & & = & 12 & (2) \\ & & -4x_2 & & +x_4 & -x_5 & = & -12 & (3') = (3) - (2) \end{array}$$

Now x_2 should enter. The row ratios are $\frac{4}{2}, \frac{12}{3}, \frac{12}{4}$: so we choose x_3 to leave. This corresponds to the point where $x_3 = x_4 = 0$, $x_1 = 6$, $x_2 = 2$, $x_5 = 4$, which is feasible. Eliminating leads to

$$\begin{array}{cccccccl}
2C & & & -5x_3 & +x_4 & & = & 28 \qquad (C'') = 2(C') + 5(1') \\
& & -2x_2 & -x_3 & +x_4 & & = & -4 \qquad (1') \\
& 2x_1 & & -3x_3 & +x_4 & & = & 12 \qquad (2') = 2(2) + 3(1') \\
& & & 2x_3 & \boxed{-x_4} & -x_5 & = & -4 \qquad (3'') = (3') - 2(1')
\end{array}$$

Now x_4 should enter the basis. Row ratios suggest that either x_2 or x_5 should leave. The first possibility would be to reverse the last step: so x_5 leaves. This brings us to the point where $x_3 = x_5 = 0$ and $x_1 = x_2 = x_4 = 4$: a feasible corner. We find

$$\begin{array}{cccccccl}
2C & & & -3x_3 & & -x_5 & = & 24 \qquad (C''') = (C'') + (3'') \\
& & & & & & & \\
& & -2x_2 & +x_3 & & -x_5 & = & -8 \qquad (1'') = (1') + (3'') \\
& 2x_1 & & -x_3 & & -x_5 & = & 8 \qquad (2'') = (2') + (3'') \\
& & & 2x_3 & -x_4 & -x_5 & = & -4 \qquad (3'')
\end{array}$$

This C-equation shows now that C attains its minimum value $\dfrac{24}{2} = 12$ at the point $(4, 4)$.

There was no need to find $(1'')$ and $(2'')$.

Graphical solution