Part 2

1 Linear Equations in Modelling

1.1 Finite difference method

Meshes & nodes

Suppose that we are given the numerical values

$$y_j = f(x_j) \qquad (0 \leq j \leq n)$$

of a function $y = f(x)$ at the *nodes*

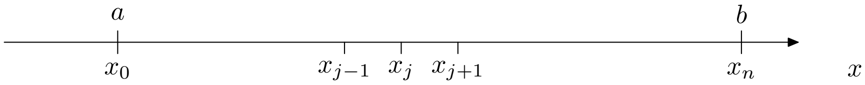
$$x_j \ (0 \leq j \leq n)$$

of a *uniform mesh* on an interval $[a, b]$.

The nodes are

$$x_j = a + jh = a + n\frac{b-a}{n} :$$

they divide the original interval $[a, b]$ into n equal subintervals $[x_{j-1}, x_j]$ $(1 \leq j \leq n)$, each of length $h = \dfrac{b-a}{n}$.



$$h \ = \ x_j - x_{j-1} \ = \frac{b-a}{n}$$

HOW MIGHT WE ESTIMATE y' AND y'' FROM THESE DATA ?

The obvious approximation to y'_j $[= f'(x_j)]$ *from the right* is

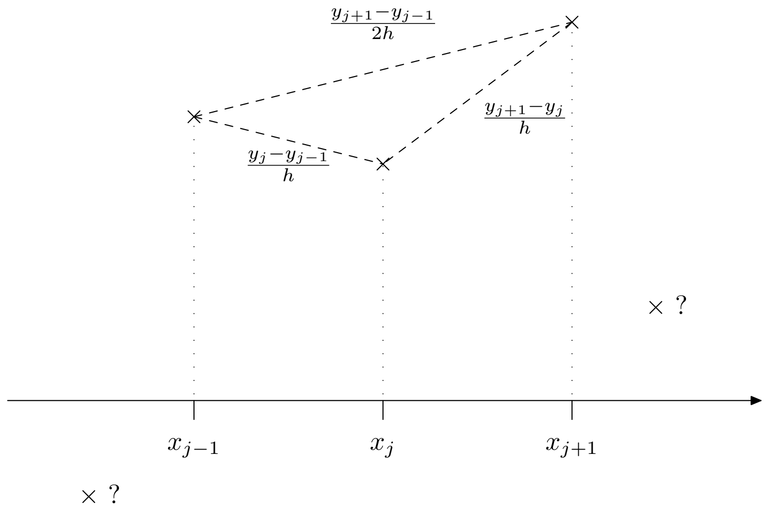
$$y'_{j,r} = \frac{y_{j+1} - y_j}{h} \qquad (0 \le j < n)$$

while *from the left* the obvious approximation to y'_j is

$$y'_{j,l} = \frac{y_j - y_{j-1}}{h} \qquad (0 < j \le n)$$

If we *average* these we get the *central approximation*

$$y'_{j,c} = \frac{y_{j+1} - y_{j-1}}{2h} \qquad (0 < j < n)$$



We can estimate the *second derivatives* by taking difference quotients using the *estimated* first derivatives — using $y'_{j,r}$ on the right and $y'_{j,l}$ on the left:

$$y''_j \simeq \left\{ \frac{y_{j+1} - y_j}{h} - \frac{y_j - y_{j-1}}{h} \right\} \frac{1}{h} = \frac{y_{j-1} - 2y_j + y_{j+1}}{h^2} \qquad (0 < j < n)$$

These are called the *central difference formulae* for the first and second derivatives.

HOW GOOD DO WE EXPECT THESE ESTIMATES TO BE ?

Taylor's Theorem & the (central) difference formulae

Taylor's Theorem states that at each x, for h sufficiently small, we have

$$f(x+h) \quad = \quad f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$$

and that

$$f(x-h) \quad = \quad f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \cdots$$

Subtracting gives

$$f(x+h) - f(x-h) = 2hf'(x) + 2\frac{h^3}{3!}f'''(x) + \cdots$$

so that

$$f'(x) \simeq \frac{f(x+h) - f(x-h)}{2h} + O(h^2).$$

Adding gives

$$f(x+h) + f(x-h) = 2f(x) + 2\frac{h^2}{2!}f''(x) + 2\frac{h^4}{4!}f''''(x) + \cdots$$

so that

$$f''(x) \simeq \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} + O(h^2).$$

When $x = x_j$, $h = \dfrac{b-a}{n}$, we get

$$f''(x_j) \simeq \frac{y_{j-1} - 2y_j + y_{j+1}}{h^2} + O(h^2)$$

so

THE CENTRAL DIFFERENCE FORMULAE ARE ACCURATE TO ORDER h^2

At endpoints

Taking linear combinations of the above Taylor expansions leads to the approximation

$$f'(x) \simeq \frac{4f(x+h) - f(x+2h) - 3f(x)}{2h} + O(h^2)$$

in terms of the values of f at x, $x+h$ and $x+2h$, and the approximation

$$f'(x) \simeq \frac{f(x-2h) + 3f(x) - 4f(x-h)}{2h} + O(h^2)$$

in terms of the values of f at x, $x-h$ and $x-2h$.

In mesh notation

$$\begin{array}{lclll} y_1 & \simeq & y_0 & +hy_0' & +\frac{h^2}{2!}y_0'' \\ y_2 & \simeq & y_0 & +2hy_0' & +4\frac{h^2}{2!}y_0'' \end{array}$$

so

$$-4y_1 + y_2 \simeq -3y_0 - 2hy_0' + O(h^3)$$

from which we get the *left endpoint* approximation

$$\begin{aligned} y_0' &\simeq \frac{-3y_0 + 4y_1 - y_2}{2h} \\ &= 2\frac{-y_0 + y_1}{h} - \frac{-y_0 + y_2}{2h} \qquad \textit{Interpretation?} \end{aligned}$$

while at the *right endpoint* we use the approximation

$$\begin{aligned} y_n' &\simeq \frac{y_{n-2} - 4y_{n-1} + 3y_n}{2h} \\ &= 2\frac{-y_{n-1} + y_n}{h} - \frac{-y_{n-2} + y_n}{2h} \end{aligned}$$

E Use the difference formulae to estimate $\frac{d}{dx}(\sin x)$ at each mesh point, from the data

x	0.5	0.6	0.7
$\sin x$	0.47943	0.56464	0.64422

.

S Using the mesh on $[0.5, 0.7]$ with $h = 0.1$, so that $x_0 = 0.5$, $x_1 = 0.6$, $x_2 = 0.7$, we obtain:

using the right hand difference formula at the left end-point,

$$\begin{aligned} \left.\frac{d}{dx}(\sin x)\right|_{x=0.5} &\simeq \frac{4\sin(0.5+0.1) - \sin(0.5+0.2) - 3\sin(0.5)}{2(0.1)} \\ &\simeq \frac{4(0.56464) - 0.64422 - 3(0.47943)}{2(0.1)} \\ &\simeq 0.88052; \end{aligned}$$

using the central difference formula at 0.6,

$$\begin{aligned} \left.\frac{d}{dx}(\sin x)\right|_{x=0.6} &\simeq \frac{\sin(0.7) - \sin(0.5)}{2(0.1)} \\ &\simeq 0.82395; \end{aligned}$$

and using the left hand difference formula at the right end-point,

$$\begin{aligned} \left.\frac{d}{dx}(\sin x)\right|_{x=0.7} &\simeq \frac{3(0.64422) + 0.47943 - 4(0.56464)}{2(0.1)} \\ &\simeq 0.76765. \end{aligned}$$

In tabular form

x	0.5	0.6	0.7
$\frac{d}{dx}\sin x$ (*estimated*)	0.88052	0.82395	0.76765
actual value	0.87758	0.82533	0.76484

E Use the difference formulae to estimate $\dfrac{d}{dx}(\sin x)$ at each mesh point, from the data

x	0.5	0.55	0.6	0.65	0.7
$\sin x$	0.479426	0.522687	0.564642	0.605186	0.644218

S Using the mesh on $[0.5, 0.7]$ with $h = 0.05$, so that $x_0 = 0.5$, $x_1 = 0.55$, $x_2 = 0.6$, $x_3 = 0.65$, $x_4 = 0.7$ we obtain:

using the right hand difference formula at the left end-point,

$$\begin{aligned}\left.\frac{d}{dx}(\sin x)\right|_{x=0.5} &\simeq \frac{4\sin(0.5+0.05) - \sin(0.5+0.1) - 3\sin(0.5)}{2(0.05)} \\ &\simeq \frac{4(0.522687) - 0.564642 - 3(0.479426)}{0.1} \\ &\simeq 0.878298;\end{aligned}$$

using the central difference formula at 0.6,

$$\begin{aligned}\left.\frac{d}{dx}(\sin x)\right|_{x=0.6} &\simeq \frac{\sin(0.65) - \sin(0.55)}{2(0.05)} \\ &\simeq 0.824992;\end{aligned}$$

etc.

In tabular form

x	0.5	0.55	0.6	0.65	0.7
$\dfrac{d}{dx}\sin x$ (*estimated*)	0.878298	0.852169	0.824992	0.795752	0.765499
actual value	0.877583	0.877583	0.825336	0.796084	0.764842

By spreadsheet

formulae

	x	$\sin(x)$	*estimated derivative*	$\cos(x)$
$A1$	0.5	$= SIN(A1)$	$= (4 * B2 - B3 - 3 * B1)/(2 * (A2 - A1))$	$= COS(A1)$
$A2$	0.55	$= SIN(A2)$	$= (B3 - B1)/(2 * (A2 - A1))$	$= COS(A2)$
$A3$	0.6	$= SIN(A3)$	$= (B4 - B2)/(2 * (A3 - A2))$	$= COS(A3)$
$A4$	0.65	$= SIN(A4)$	$= (B5 - B3)/(2 * (A4 - A3))$	$= COS(A4)$
$A5$	0.7	$= SIN(A5)$	$= (B3 + 3 * B5 - 4 * B4)/(2 * (A5 - A4))$	$= COS(A5)$

values

	x	$\sin(x)$	*estimated derivative*	$\cos(x)$
$A1$	0.5	0.479425539	0.878298265	0.877582562
$A2$	0.55	0.522687229	0.852169348	0.852524522
$A3$	0.6	0.564642473	0.824991768	0.825335615
$A4$	0.65	0.605186406	0.795752138	0.796083799
$A5$	0.7	0.644217687	0.765499122	0.764842187

1.2 Solving *ODE*s

Suppose we seek a numerical approximation to the solution $y = f(x)$ of an *ODE* on an interval $[a, b]$. We choose a suitable [largish] integer n and set up the uniform n-mesh on this interval with nodes $x_j = a + jh = a + n\dfrac{b-a}{n}$ $(0 \le j \le n)$.

We write $y_j = f(x_j)$ $(0 \le j \le n)$.

The general *linear second order boundary value problem* has the form

$$\begin{aligned} &a(x)y'' + b(x)y' + c(x)y = d(x) \qquad\qquad (a \le x \le b)\\ &\alpha y'(a) + \beta y(a) = p,\\ &\gamma y'(b) + \delta y(b) = q \end{aligned}$$

where $a(x)$, $b(x)$, $c(x)$ and $d(x)$ are known functions and α, β, γ, δ, p and q are known constants.

Applying the central difference formulae at the $n-1$ internal nodes provides us with $n-1$ equations towards determining the $n+1$ quantities y_j.

The right hand difference formula and the left boundary condition provide one more equation: the left hand difference formula and the right boundary condition provide another — so we have $n+1$ equations in all.

E Find the solution of the boundary value problem

$$y'' + y = 12x, \qquad\qquad y(0) = 0, \quad y(1) = 6,$$

using the dissection of $[0, 1]$ with $h = \frac{1}{4}$.

S Here we have 5 nodes: $x_j = \frac{j}{4}$ $(0 \le j \le 4)$. The 5 unknown values are $y_j = y(\frac{j}{4})$ $(0 \le j \le 4)$. The boundary conditions immediately give us

$$y_0 = y(0) = 0, \qquad y_4 = y(1) = 6.$$

At the interior nodes we use the [approximations given by the] central difference formulae for the second derivative (the first derivative does not feature in this example):

$$y''(x_j) \simeq \frac{y_{j-1} - 2y_j + y_{j+1}}{h^2} = 16[y_{j-1} - 2y_j + y_{j+1}].$$

Now the *ODE* is

$$y'' + y = 12x,$$

so, for $j = 1, 2, 3$,

$$16(y_{j-1} - 2y_j + y_{j+1}) + y_j = 12x_j$$

and therefore

$$\begin{array}{rrrrrcrcr} 16y_0 & -31y_1 & +16y_2 & & & = & 12\left(\frac{1}{4}\right) & = & 3 \\ & 16y_1 & -31y_2 & +16y_3 & & = & 12\left(\frac{1}{2}\right) & = & 6 \\ & & 16y_2 & -31y_3 & +16y_4 & = & 12\left(\frac{3}{4}\right) & = & 9. \end{array}$$

Remembering that $y_0 = 0$ and $y_4 = 6$ we get

$$\begin{array}{rrrcll} -31y_1 & +16y_2 & & = & 3 & \\ 16y_1 & -31y_2 & +16y_3 & = & 6 & \\ & 16y_2 & -31y_3 & = & 9 - 6 \times 16 & = -87 \end{array}$$

Solve using Gaussian elimination [working to 2 dp]:

$$\begin{aligned} \begin{bmatrix} -31 & 16 & 0 & 3 \\ 16 & -31 & 16 & 6 \\ 0 & 16 & -31 & -87 \end{bmatrix} & \sim \begin{bmatrix} -31 & 16 & 0 & 3 \\ 0 & -22.74 & 16 & 7.55 \\ 0 & 16 & -31 & -87 \end{bmatrix} \\ & \sim \begin{bmatrix} -31 & 16 & 0 & 3 \\ 0 & -22.74 & 16 & 7.55 \\ 0 & 0 & -19.74 & -81.68 \end{bmatrix}. \end{aligned}$$

Back substitution gives $y_3 = 4.14$, $y_2 = 2.58$, $y_1 = 1.23$.

The solution of this problem can be derived analytically. It is

$$y = 12x - 6\frac{\sin x}{\sin(1)},$$

so we can assess the accuracy of our approximate solution:

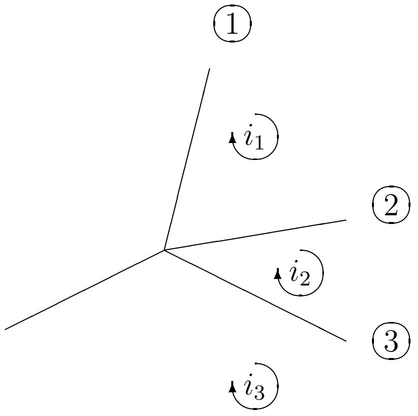
j	x_j	*estimated* y_j *(2dp)*	*known* y_j *(3dp)*
1	$\frac{1}{4}$	1.23	1.236
2	$\frac{2}{4}$	2.58	2.582
3	$\frac{3}{4}$	4.14	4.140

1.3 Network problems

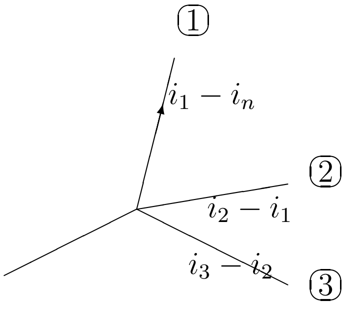
Networks describe 'flows under forces': *eg* fluid flow around pipework, flow of electricity around a circuit, cars moving around the road network.

All flows satisfy two properties:
(i) Flow is *conserved at junctions* [what goes in must come out].
(ii) The *flow* is related to the *force*: *eg* Ohm's Law.

To satisfy the first condition we use the idea of *loop flows*.



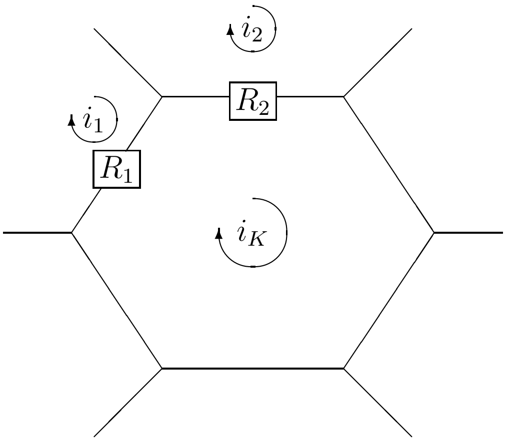
From them we compute the flow in each branch:



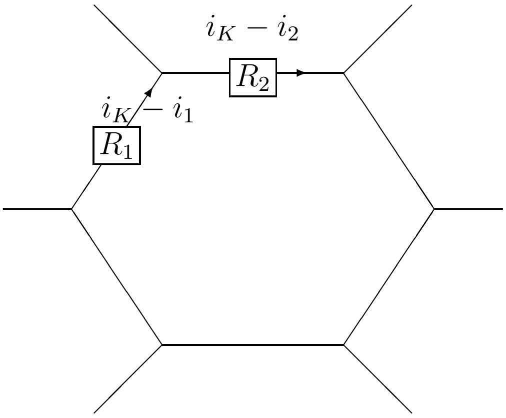
Total flow out of junction $= i_1 - i_n + i_2 - i_1 + \ldots + (i_n - i_{n-1}) = 0$.

Closed loops & currents

Loop currents



combine to give *net currents* in the branches

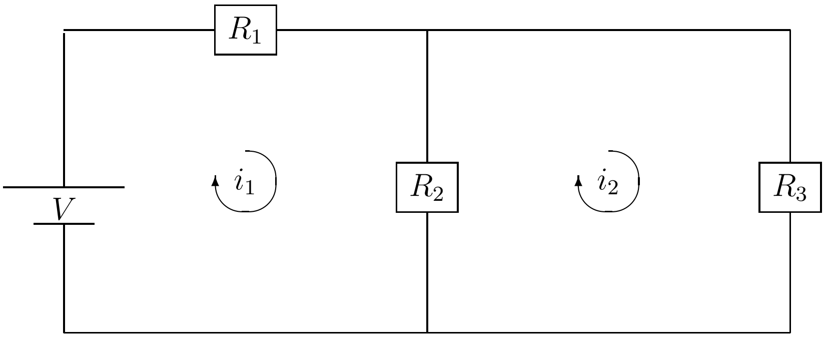


The forces around the closed loop are

$$R_1(i_K - i_1) + R_2(i_K - i_2) + \dots R_n(i_K - i_n).$$

This must equal the force applied around the circuit *eg* by batteries, pump, *etc.*

E A known voltage V is applied to the circuit shown.



What currents flow in the wires?

S The loop equations are

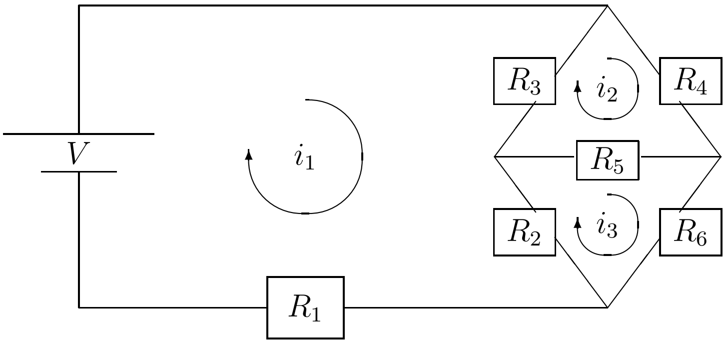
$$\begin{array}{lcl}R_1 i_1 + R_2(i_1 - i_2) = V & \qquad & \textit{loop } 1 \\ R_3 i_2 + R_2(i_2 - i_1) = 0 & \qquad & \textit{loop } 2\end{array}$$

In matrix form:

$$\begin{bmatrix} R_1 + R_2 & -R_2 \\ -R_2 & R_2 + R_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V \\ 0 \end{bmatrix}$$

Note that loop analysis always leads to a *symmetric* matrix.

Wheatstone Bridge



The loop equations are

$$\begin{array}{ccccccccc} R_1 i_1 & + & R_3(i_1 - i_2) & + & R_2(i_1 - i_3) & = & V & & \textit{loop } 1 \\ R_4 i_2 & + & R_5(i_2 - i_3) & + & R_3(i_2 - i_1) & = & 0 & & \textit{loop } 2 \\ R_6 i_3 & + & R_2(i_3 - i_1) & + & R_5(i_3 - i_2) & = & 0 & & \textit{loop } 3 \end{array}$$

ie

$$\begin{bmatrix} R_1 + R_2 + R_3 & -R_3 & -R_2 \\ -R_3 & R_3 + R_4 + R_5 & -R_5 \\ -R_2 & -R_5 & R_2 + R_5 + R_6 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} V \\ 0 \\ 0 \end{bmatrix}$$

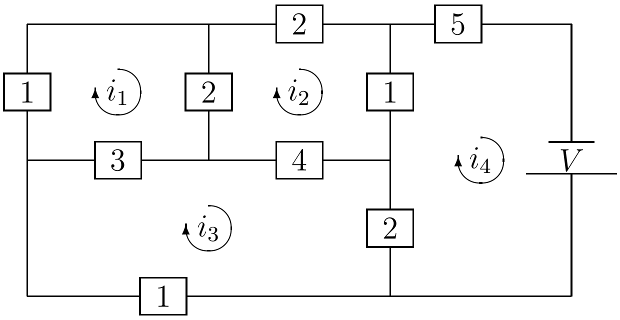
The bridge is said to be *balanced* when no current flows through R_5 *ie* when $i_2 = i_3$. Substituting this condition in the second and third loop equations gives

$$\begin{array}{rcl} R_4 i_2 + R_3(i_2 - i_1) & = & 0 \\ R_6 i_2 + R_2(i_2 - i_1) & = & 0 \end{array}$$

ie

$$R_2\, R_4 = R_3\, R_6$$

E Solve the network so as to find its resistance to the source V.



S Here

$$\begin{array}{llllllll}
i_1 & + & 2(i_1 - i_2) & + & 3(i_1 - i_3) & & & = & 0 & \textit{loop } 1 \\
2i_2 & + & (i_2 - i_4) & + & 4(i_2 - i_3) & + & 2(i_2 - i_1) & = & 0 & \textit{loop } 2 \\
i_3 & + & 3(i_3 - i_1) & + & 4(i_3 - i_2) & + & 2(i_3 - i_4) & = & 0 & \textit{loop } 3 \\
5i_4 & + & 2(i_4 - i_3) & + & (i_4 - i_2) & & & = & V & \textit{loop } 4
\end{array}$$

ie

$$\begin{bmatrix} 6 & -2 & -3 & 0 \\ -2 & 9 & -4 & -1 \\ -3 & -4 & 10 & -2 \\ 0 & -1 & -2 & 8 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ V \end{bmatrix}$$

Now

$$\begin{bmatrix} 6 & -2 & -3 & 0 & 0 \\ -2 & 9 & -4 & -1 & 0 \\ -3 & -4 & 10 & -2 & 0 \\ 0 & -1 & -2 & 8 & V \end{bmatrix} \sim \begin{bmatrix} 6 & -2 & -3 & 0 & 0 \\ 0 & \frac{25}{3} & -5 & -1 & 0 \\ 0 & -5 & \frac{17}{2} & -2 & 0 \\ 0 & -1 & -2 & 8 & V \end{bmatrix}$$

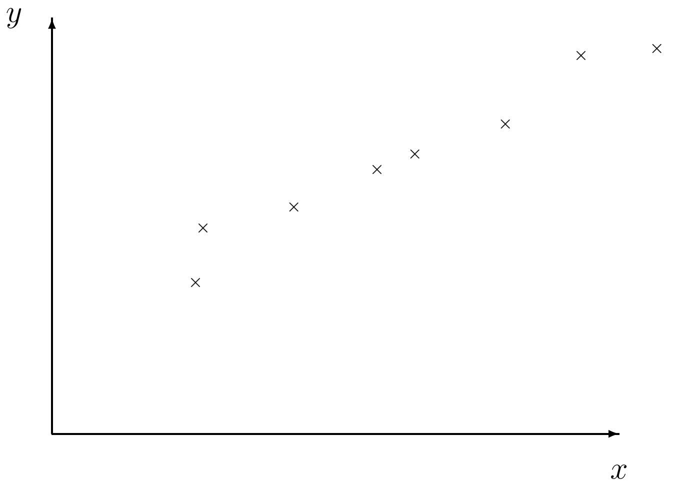
$$\sim \begin{bmatrix} 6 & -2 & -3 & 0 & 0 \\ 0 & \frac{25}{3} & -5 & -1 & 0 \\ 0 & 0 & \frac{11}{2} & -\frac{13}{5} & 0 \\ 0 & 0 & -\frac{13}{5} & \frac{197}{5} & V \end{bmatrix}$$

$$\sim \begin{bmatrix} 6 & -2 & -3 & 0 & 0 \\ 0 & \frac{25}{3} & -5 & -1 & 0 \\ 0 & 0 & \frac{11}{2} & -\frac{13}{5} & 0 \\ 0 & 0 & 0 & \frac{1829}{275} & V \end{bmatrix}$$

The resistance of the network is $\dfrac{V}{i_4} = \dfrac{1829}{275}$.

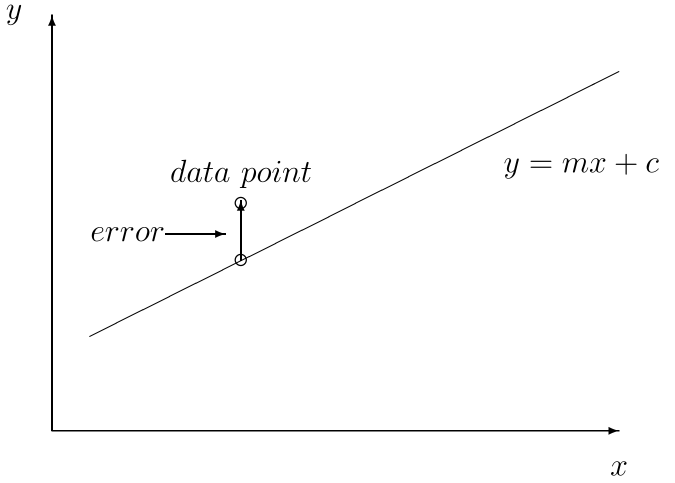
1.4 Least squares fitting

Given a set of *data points* (x_j, y_j) $(1 \leq j \leq n)$ we may seek the 'best' straight line $y = mx + c$ which fits them [or, that they fit to].



Each data point has its own 'error'

$$E_j = |\, y(x_j) - y_j \,| = |\, mx_j + c - y_j \,| \,.$$



The sum $\sum_{j=1}^{n} E_j$ is a natural measure of the 'fit' of the line to the data. The problem of finding m and c to minimise this is, however, intractable.

Instead we combine these individual errors by adding their squares, to get the expression

$$E = \sum_{j=1}^{n} E_j^2 = \sum_{j=1}^{n} |\, mx_j + c - y_j \,|^2 = \sum_{j=1}^{n} (mx_j + c - y_j)^2$$

and seek the values of m and c which will minimise *this* total.

To do this we may treat m and c as independent variables [which they are], and look to see where

$$\frac{\partial}{\partial m} E = \sum_{j=1}^{n} 2(mx_j + c - y_j)x_j = 0,$$

and

$$\frac{\partial}{\partial c} E = \sum_{j=1}^{n} 2(mx_j + c - y_j) = 0.$$

Normal equations

These last two equations can be rewritten in the form

$$\begin{aligned} m\sum_{j=1}^{n} x_j^2 + c\sum_{j=1}^{n} x_j &= \sum_{j=1}^{n} x_j y_j \\ m\sum_{j=1}^{n} x_j + cn &= \sum_{j=1}^{n} y_j, \end{aligned}$$

and are called the *normal equations* for m and c.

If we write

$$\alpha = \sum x_j^2, \quad \beta = \sum x_j, \quad \gamma = \sum x_j y_j, \quad \delta = \sum y_j$$

we can restate these equations as

$$\begin{aligned} \alpha m + \beta c &= \gamma \\ \beta m + nc &= \delta. \end{aligned}$$

Alternative derivation of the normal equations

For brevity write

$$\zeta = \sum y_j^2$$

We have

$$\begin{aligned}
\alpha E &= \alpha \sum (mx_j + c - y_j)^2 \\
&= \alpha^2 m^2 + 2\alpha(\beta c - \gamma)m + \alpha \left\{nc^2 + \zeta - 2\delta c\right\} \\
&= \{\alpha m + \beta c - \gamma\}^2 + (\alpha n - \beta^2)c^2 + 2(\beta\gamma - \alpha\delta)c - \gamma^2 + \alpha\zeta \\
&= \{\alpha m + \beta c - \gamma\}^2 + (\alpha n - \beta^2)\left\{c^2 + 2\frac{\beta\gamma - \alpha\delta}{\alpha n - \beta^2}c\right\} - \gamma^2 + \alpha\zeta \\
&= \{\alpha m + \beta c - \gamma\}^2 + (\alpha n - \beta^2)\left\{c + \frac{\beta\gamma - \alpha\delta}{\alpha n - \beta^2}\right\}^2 + \alpha\zeta - \frac{(\beta\gamma - \alpha\delta)^2}{\alpha n - \beta^2} - \gamma^2
\end{aligned}$$

Thus E is minimised when m and c satisfy

$$\alpha m + \beta c = \gamma$$

and

$$(\alpha n - \beta^2)c + \beta\gamma - \alpha\delta = 0$$

from which

$$\alpha nc - \alpha\delta = \beta^2 c - \beta\gamma = \beta[\beta c - \gamma] = -\alpha\beta m$$

and thus

$$\beta m + nc = \delta$$

as before.

Substituting these values —

$$E_{min} = \zeta - \frac{1}{\alpha}\left\{\frac{(\beta\gamma - \alpha\delta)^2}{\alpha n - \beta^2} + \gamma^2\right\}$$

Matrix formulation for least squares fit

Let

$$A = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \\ x_n & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \text{and} \quad Z = \begin{bmatrix} m \\ c \end{bmatrix}.$$

Then

$$A^{\mathrm{T}} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \\ 1 & 1 & \dots & 1 \end{bmatrix},$$

so that

$$A^{\mathrm{T}} A = \begin{bmatrix} \sum_{j=1}^{n} x_j^2 & \sum_{j=1}^{n} x_j \\ \sum_{j=1}^{n} x_j & n \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \beta & n \end{bmatrix} \quad \text{and} \quad A^{\mathrm{T}} B = \begin{bmatrix} \sum_{j=1}^{n} x_j y_j \\ \sum_{j=1}^{n} y_j \end{bmatrix} = \begin{bmatrix} \gamma \\ \delta \end{bmatrix}.$$

Thus the normal equations are

$$\boxed{A^{\mathrm{T}} A\, Z = A^{\mathrm{T}} B.}$$

E Suppose we have the data

x	0	1	1	2	2
y	0	0	1	2	3

.

What is the best straight line that fits this data in the least squares sense?

S Here

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 1 \\ 2 & 1 \\ 2 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 3 \end{bmatrix},$$

so that

$$A^{\mathrm{T}}A = \begin{bmatrix} 0 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 1 \\ 2 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 5 \end{bmatrix},$$

and

$$A^{\mathrm{T}}B = \begin{bmatrix} 0 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 6 \end{bmatrix}.$$

The normal equations are

$$\begin{bmatrix} 10 & 6 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} 11 \\ 6 \end{bmatrix}$$

and their solutions are

$$\begin{bmatrix} m \\ c \end{bmatrix} = \frac{1}{50-36} \begin{bmatrix} 5 & -6 \\ -6 & 10 \end{bmatrix} \begin{bmatrix} 11 \\ 6 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 55-36 \\ -66+60 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 19 \\ -6 \end{bmatrix}.$$

The line of best fit is therefore

$$y = \frac{19}{14}x - \frac{3}{7}.$$

Remember that, if $ad - bc \neq 0$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}:$$

(see *Cramer's Rule*, below).

OR, *neater*, start from $A^T[A : B]$ —

$$A^{\mathrm{T}}[A : B] \;=\; \begin{bmatrix} 0 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$
\begin{aligned}
&= \begin{bmatrix} 10 & 6 & 11 \\ 6 & 5 & 6 \end{bmatrix} \\
&\sim \begin{bmatrix} 30 & 18 & 33 \\ 6 & 5 & 6 \end{bmatrix} \\
&\sim \begin{bmatrix} 0 & -7 & 3 \\ 6 & 5 & 6 \end{bmatrix} \\
&\sim \begin{bmatrix} 0 & -7 & 3 \\ 42 & 0 & 57 \end{bmatrix}
\end{aligned}
$$

from which $c = -\frac{3}{7}$ and then $m = \frac{57}{42} = \frac{19}{14}$.

With the notation as above in this example we have

$$\alpha = 10, \quad \beta = 6, \quad \gamma = 11, \quad \delta = 6, \quad \zeta = 14, \quad n = 5.$$

Then

$$\alpha n - \beta^2 = 14, \quad \beta\gamma - \alpha\delta = 6$$

and so

$$E_{min} = \frac{23}{14}$$

as may be checked directly.

Other polynomial approximations can be treated similarly

E Given the data

x	1	1.1	1.3	1.5	1.9	2.1
y	1.85	1.96	2.20	2.50	2.97	3.21

find the best fit by a quadratic polynomial of the form $y = ax^2 + bx + c$.

S The normal equations still have the form

$$A^{\mathrm{T}} A\, Z = A^{\mathrm{T}} B\,;$$

but now

$$A = \begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ \vdots & & \\ x_n^2 & x_n & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \text{and} \quad Z = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Here

$$A = \begin{bmatrix} 1^2 & 1 & 1 \\ 1.1^2 & 1.1 & 1 \\ \vdots & \vdots & \vdots \\ 2.1^2 & 2.1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1.85 \\ 1.96 \\ \vdots \\ 3.21 \end{bmatrix}$$

and the solution is $a \simeq -0.0510$, $b \simeq 1.4077$ and $c \simeq 0.4807$.