Part 1

1 Systems of Linear Equations

1.1 Matrices

Given A and $B \in M_{m\times n}(\mathbf{F})$ and $\lambda \in \mathbf{F}$ [where $\mathbf{F}$ is either $\mathbf{R}$ or $\mathbf{C}$] define

$$\lambda A = [\lambda\, a_{ij}]\,, \quad A + B = [a_{ij} + b_{ij}]\,.$$

E

$$2\begin{bmatrix} 0 & -2 & 7 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -4 & 14 \\ -2 & 2 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 7 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 10 \\ 3 & 6 & 9 \end{bmatrix}.$$

Write $\mathbf{F}_n$ for the space of *row vectors* $M_{1\times n}(\mathbf{F})$ and $\mathbf{F}^n$ for the space of *column vectors* $M_{n\times 1}(\mathbf{F})$.

Given $A \in M_{m\times n}(\mathbf{F})$ and $B \in M_{n\times p}(\mathbf{F})$ define $C = [c_{ij}] \in M_{m\times p}(\mathbf{F})$ by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}\,.$$

E Let

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \\ 5 & 6 \end{bmatrix} \qquad \textit{and} \qquad B = \begin{bmatrix} -1 & 2 & 1 \\ 0 & -7 & 1 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 1 & 2 \\ -3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 2 & 1 \\ 0 & -7 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -12 & 3 \\ 3 & -34 & 1 \\ -5 & -32 & 11 \end{bmatrix}.$$

Motivation for definition of multiplication

Suppose we have quantities u, v specified in a linear manner in terms of x, y, z: *eg*

$$\begin{aligned} u &= -x + 2y + z \\ v &= - 7y + z \end{aligned}$$

and that then a, b, c are defined linearly in terms of u, v: *eg*

$$\begin{aligned} a &= u + 2v \\ b &= -3u + 4v \\ c &= 5u + 6v \end{aligned}$$

Direct substitution enables us to express the variables a, b, c directly in terms of the x, y, z:

$$\begin{aligned} a &= u + 2v \\ &= -x + 2y + z + 2(-7y + z) \\ &= -x - 12y + 3z \end{aligned}$$

$$\begin{aligned} b &= -3u + 4v \\ &= -3(-x + 2y + z) + 4(-7y + z) \\ &= 3x - 34y + z \end{aligned}$$

$$\begin{aligned} c &= 5u + 6v \\ &= 5(-x + 2y + z) + 6(-7y + z) \\ &= -5x - 32y + 11z \end{aligned}$$

In matrix terminology, with A and B as above,

$$\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix} &= \begin{bmatrix} -1 & 2 & 1 \\ 0 & -7 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = B \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ -3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = A \begin{bmatrix} u \\ v \end{bmatrix} \end{aligned}$$

It is natural to write

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = A \begin{bmatrix} u \\ v \end{bmatrix} = A \left(B \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = A \circ B \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and we have calculated that

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 & -12 & 3 \\ 3 & -34 & 1 \\ -5 & -32 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = AB \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The composition of the mappings specified by the matrices A and B corresponds to the matrix product AB

Properties of matrix multiplication

Matrix multiplication is *associative*:

$$A(BC) = (AB)C$$

whenever *either* side is defined.

When $AB = BA$ we say that A and B *commute.* In general at most *one* of AB and BA will be defined: and even if both are, usually $AB \neq BA$.

Transpose of a matrix

Given $A \in M_{m\times n}(\mathbf{F})$ we define its *transpose* $A^{\mathrm{T}} \in M_{n\times m}(\mathbf{F})$ as the matrix whose ij^{th} entry is a_{ji}, *eg*

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

The rows of A become the columns of A^{T} [and *vice versa*]. There is a *reversal rule*

$$(AB)^T = B^T A^T$$

Block multiplication

We may *partition* a matrix into *submatrices*, or *blocks* — *eg*

$$A = \begin{bmatrix} 3 & -4 & 1 & \vdots & 0 & 2 \\ -1 & 5 & -3 & \vdots & 1 & 4 \\ \cdots & \cdots & \cdots & & \cdots & \cdots \\ 2 & 0 & -2 & \vdots & 1 & 6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

If two matrices are partitioned compatibly we may multiply the block matrices as if they are ordinary matrices. *Eg* if

$$B = \begin{bmatrix} 2 & -1 \\ 3 & 0 \\ -5 & 1 \\ \cdots & \cdots \\ 4 & -3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}$$

then

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} \end{bmatrix} = \begin{bmatrix} -11 & 2 \\ 32 & 3 \\ 18 & 5 \end{bmatrix}$$

as you may CHECK.

Identity & zero matrices

The $m \times n$ *zero matrix* O_{mn} has all its entries equal to zero. The $n \times n$ *identity matrix* I_n has its jj^{th} entry equal to 1 $(1 \le j \le n)$ while the other entries are zeros.

Invertibility

An $n \times n$ *square matrix* A is *invertible* if there is some other matrix P such that $AP = PA = I_n$. Such a matrix P is called the *inverse* of A: we usually denote it by A^{-1}.

If A and B (square of the same size) are both invertible then so is AB and [*Reversal Rule*]

$$(AB)^{-1} = B^{-1}\,A^{-1}$$

More generally, any product of invertibles is invertible.

Remark. If A has a zero row [or a zero column] then A is *singular*, *ie* not invertible.

Remark. A is invertible if and only if A^T is invertible: and if so

$$(A^T)^{-1} = (A^{-1})^T$$

(Reduced) echelon form

A matrix A is in *echelon form* if

1. Beneath a row containing only zeroes there are no non-zero entries
2. The leading entry of any given row is to the right of the leading entry of all rows above the given row
3. In a column containing a leading entry all entries below the leading entry are zero

A matrix A is in *reduced echelon form* if, additionally,

1. The leftmost non-zero entry in a row [if any] is a 1 (the leading 1 of the row)
2. In a column containing a leading 1 all entries other than the leading 1 are zeroes.

Theorem. Suppose B is a square matrix in reduced echelon form. Then B is invertible if and only if $B = I$.

1.2 *Ero*s & their matrices

Elementary row operations & elementary matrices

There are three kinds of *ero* and to each there corresponds the *elementary matrix* obtained by applying it to an identity matrix : —

(i) Interchanging rows

(ii) Multiplying a row by a nonzero constant

(iii) Adding a multiple of one row to another row

For instance

(i)

$$\begin{bmatrix} -1 & -12 & 3 \\ 3 & -34 & 1 \\ -5 & -12 & 11 \end{bmatrix} \sim \begin{bmatrix} -1 & -12 & 3 \\ -5 & -12 & 11 \\ 3 & -34 & 1 \end{bmatrix} \quad R_2 \longleftrightarrow R_3 \ .$$

The corresponding elementary matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Such matrices are often called *permutation matrices.*

Permutation matrices are *self-inverse*: if P is a permutation matrix then $P^2 = I$, so $P^{-1} = P$.

(ii)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ -12 & -15 & -18 \\ 7 & 8 & 9 \end{bmatrix} \quad R_2 \rightarrow -3R_2$$

(iii)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 7 & 8 & 9 \end{bmatrix} \quad R_2 \to R_2 - 2R_1$$

Note that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -12 & 3 \\ 3 & -34 & 1 \\ -5 & -12 & 11 \end{bmatrix} = \begin{bmatrix} -1 & -12 & 3 \\ -5 & -12 & 11 \\ 3 & -34 & 1 \end{bmatrix} \qquad (i)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ -12 & -15 & -18 \\ 7 & 8 & 9 \end{bmatrix} \qquad (ii)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 7 & 8 & 9 \end{bmatrix}. \qquad (iii)$$

In all cases

left multiplying by the corresponding elementary matrix
is equivalent to performing the ero*.*

In a self-explanatory notation

$$E_{R_i\longleftrightarrow R_j} \;=\; \begin{array}{cl} \begin{bmatrix} 1 & 0 & & & \cdots & & & & \cdots & 0 \\ \vdots & & & & & & & & & \\ 0 & & & & & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & & & & & \\ 0 & \cdots & 0 & 1 & 0 & & \cdots & & & 0 \\ \vdots & & & & & & & & & \\ 0 & & & & \cdots & & & & & 1 \end{bmatrix} & \begin{array}{l} \\ \\ i^{th}\ row\ j^{th}\ column \\ \\ j^{th}\ row\ i^{th}\ column \\ \\ \\ \end{array} \end{array}$$

$$E_{R_i\rightarrow\lambda R_i} \;=\; \begin{array}{cl} \begin{bmatrix} 1 & 0 & & \cdots & & & 0 \\ & & & & & & \\ & & & \vdots & & & \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \lambda & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ & & & \vdots & & & \\ & & & & & & \\ 0 & & & \cdots & & 0 & 1 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \\ i^{th}\ row\ \&\ i^{th}\ column \\ \\ \\ \\ \\ \end{array} \end{array}$$

$$E_{R_i\rightarrow R_i+\lambda R_j} \;=\; \begin{array}{cl} \begin{bmatrix} 1 & 0 & & & & \cdots & & & 0 \\ \vdots & & & & & & & & \\ 0 & \cdots & 1 & \cdots & & \lambda & & \cdots & 0 \\ \vdots & & & & & & & & \\ 0 & \cdots & & \cdots & & & & 0 & 1 \end{bmatrix} & \begin{array}{l} i^{th}\ row: \\ 1\ in\ i^{th} column \\ \lambda\ in\ j^{th}\ column \end{array} \end{array}$$

Any *ero* can be inverted, so the associated *elementary matrices* are invertible. In fact:

$$\begin{aligned} E^{-1}_{R_i\longleftrightarrow R_j} &= E_{R_i\longleftrightarrow R_j} \\ E^{-1}_{R_i\rightarrow\lambda R_i} &= E_{R_i\rightarrow\lambda^{-1}R_i} \\ E^{-1}_{R_i\rightarrow R_i+\lambda R_j} &= E_{R_i\rightarrow R_i-\lambda R_j} \end{aligned}$$

Right multiplication by elementary matrices

Multiplication on the right by an elementary matrix will effect a corresponding *elementary column operation* —

$$\begin{bmatrix} -1 & -12 & 3 \\ 3 & -34 & 1 \\ -5 & -12 & 11 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 3 & -12 \\ -5 & 1 & -34 \\ 3 & 11 & -12 \end{bmatrix} \qquad (i)$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -6 & 3 \\ 4 & -15 & 6 \\ 7 & -24 & 9 \end{bmatrix} \qquad (ii)$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 3 \\ -6 & 5 & 6 \\ -9 & 8 & 9 \end{bmatrix}. \qquad (iii)$$

Later on we shall an example of using the same permutation matrix to multiply both on the left and right —

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 0 & 1 \\ \frac{1}{2} & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} \end{aligned}$$

Row reduction of a square matrix

Given a square $n \times n$ matrix A we can find a sequence $E_1, E_2, \ldots, E_k$ of *eros* bringing A to *reduced echelon form* B.

Let $F = E_k \cdots E_2 E_1$. Then $B = FA = E_k \cdots E_2 E_1 \, A$ and $A = F^{-1}B$.

Note that *either* both A and B are invertible *or* both A and B are singular: for F is invertible, and if A is invertible so is FA $(= B)$ while if B is invertible so is $F^{-1}B$ $(= A)$.

B may, or may not, have zero row(s). If B has a zero row it must be singular and therefore A must also be singular.

If B has no zero row then [as remarked above] $B = I$ and so $FA = I$.

Then $A = E_1^{-1} \cdots E_k^{-1}$ and $F = A^{-1}$.

Left & right invertibility

Suppose that A is *left invertible*: *ie* that there exists an L such that $LA = I$. Reduce L to reduced echelon form — $FL = B$. Then $BA = FLA = F$.

If B is singular then B has a 0 row and so F has a 0 row: impossible since F is invertible. So $B = I$. But then L is invertible and it follows that A is invertible.

Transposing, we see that A must be invertible if it is *right invertible.*

Theorem. If A is either left invertible or right invertible then A is invertible: & the inverse is uniquely determined.

For, if $LA = I$ and $AR = I$ then $L = L(AR) = (LA)R = R$.

1.3 Gauss-Jordan method

This is a method for finding the inverse of a matrix, if it has one: and of establishing the nonexistence of an inverse when there is none.

Row reduction of the augmented matrix

Given a square $n \times n$ matrix A consider the *augmented matrix* $[A : I_n]$ and perform *ero*s on this to bring A to *reduced echelon form.*

Suppose that A is indeed invertible and that $E_1, E_2, \ldots, E_k$ are the elementary matrices that correspond to the *ero*s that reduce A. Then, with F as above,

$$F\,[\,A\,:\,I_n\,] = [\,FA:\,F\,] = [\,I_n:\,F\,] = [\,I_n:\,A^{-1}\,]\,.$$

In short, the right hand part after reducing $[A : I_n]$ is A^{-1}.

Eg $A = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix}.$

The operations will annotated against the matrix obtained *after* performing them. Remember that row-labelling is *dynamic*.

$$\begin{aligned}
[A : I_n] &= \begin{bmatrix} 2 & -1 & 2 & : & 1 & 0 & 0 \\ 1 & 0 & 1 & : & 0 & 1 & 0 \\ 3 & -1 & 0 & : & 0 & 0 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & 1 & : & 0 & 1 & 0 \\ 2 & -1 & 2 & : & 1 & 0 & 0 \\ 3 & -1 & 0 & : & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 \longleftrightarrow R_2 \\ \\ \\ \end{matrix} \\
&\sim \begin{bmatrix} 1 & 0 & 1 & : & 0 & 1 & 0 \\ 0 & -1 & 0 & : & 1 & -2 & 0 \\ 0 & -1 & -3 & : & 0 & -3 & 1 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}
\end{aligned}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & : & 0 & 1 & 0 \\ 0 & 1 & 0 & : & -1 & 2 & 0 \\ 0 & -1 & -3 & : & 0 & -3 & 1 \end{bmatrix} \quad R_2 \to -R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & : & 0 & 1 & 0 \\ 0 & 1 & 0 & : & -1 & 2 & 0 \\ 0 & 0 & -3 & : & -1 & -1 & 1 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & : & 0 & 1 & 0 \\ 0 & 1 & 0 & : & -1 & 2 & 0 \\ 0 & 0 & 1 & : & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \quad R_3 \to -\tfrac{1}{3} R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 0 & : & -1 & 2 & 0 \\ 0 & 0 & 1 & : & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}. \quad R_1 \to R_1 - R_3$$

So $\qquad A^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ -1 & 2 & 0 \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}.$

The elementary matrices corresponding to the *eros* that we have used are

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}$$

$$E_7 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Direct calculation [check!] shows that

$$F = E_7\, E_6\, E_5\, E_4\, E_3\, E_2\, E_1 = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ -1 & 2 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

How to do this by spreadsheet

$A1$	2	-1	2	1	0	0	
$A2$	1	0	1	0	1	0	
$A3$	3	-1	0	0	0	1	
$A5$	1	0	1	0	1	0	*copy row*2
$A6$	2	-1	2	1	0	0	*copy row*1
$A7$	3	-1	0	0	0	1	*copy row*3
							then copy & paste – special – values (in place)
$A10$	1	0	1	0	1	0	*copy row*1
$A11$	0	-1	0	1	-2	0	$A11 = A6 - 2 * A5$ *then copy across*
$A12$	0	-1	-3	0	-3	1	$A12 = A7 - 3 * A5$ *then copy across*
							then copy & paste – special – values (in place)
$A15$	1	0	1	0	1	0	*copy row*1
$A16$	0	1	0	-1	2	0	$A16 = -A11$ *then copy across*
$A17$	0	-1	-3	0	-3	1	*copy row*3
							then copy & paste – special – values (in place)
$A20$	1	0	1	0	1	0	*copy row*1
$A21$	0	1	0	-1	2	0	*copy row*2
$A22$	0	0	-3	-1	-1	1	$A22 = A17 + A16$ *then copy across*
							then copy & paste – special – values (in place)
$A25$	1	0	1	0	1	0	*copy row*1
$A26$	0	1	0	-1	2	0	*copy row*2
$A27$	0	0	1	$0.\dot{3}$	$0.\dot{3}$	$-0.\dot{3}$	$A27 = -A22/3$ *then copy across*
							then copy & paste – special – values (in place)
$A30$	1	0	0	$-0.\dot{3}$	$0.\dot{6}$	$0.\dot{3}$	$A30 = A25 - A27$ *then copy across*
$A31$	0	1	0	-1	2	0	*copy row*2
$A32$	0	0	1	$0.\dot{3}$	$0.\dot{3}$	$-0.\dot{3}$	*copy row*3
							then copy & paste – special – values (in place)

1.4 Gaussian elimination and the LU decomposition

Gaussian elimination

This is a process for solving systems of linear equations, using *ero*s as in finding the reduced echelon form of a matrix — but without so much work.

E Consider

$$\begin{array}{rrrcr} \boxed{2}x & +y & +z & = & 5 \\ 4x & -6y & & = & -2 \\ -2x & +7y & +2z & = & 9. \end{array}$$

We do not insist on normalising the first nonzero entry in a row [to be 1]: we just perform

$$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 + R_1$$

to obtain

$$\begin{array}{rrrcr} 2x & +y & +z & = & 5 \\ & \boxed{-8}y & -2z & = & -12 \\ & 8y & +3z & = & 14. \end{array}$$

The coefficient 2 in the first equation is known as the *first pivot*.

Now work on the second & third equations. Just do

$$R_3 \to R_3 + R_2$$

and obtain

$$\begin{array}{rrrcr} 2x & +y & +z & = & 5 \\ & -8y & -2z & = & -12 \\ & & z & = & 2. \end{array}$$

The *second pivot* is the coefficient of y in the second equation *ie* -8.

This is the end of the elimination stage. We now solve by *backward substitution*: the last equation states that $z = 2$: substituting this value into the previous equation shows that $y = 1$; and substituting both these values into the first equation shows that $x = 1$.

NB Gaussian (forward) elimination produces pivots along the matrix diagonal. We continue this process until the matrix becomes *upper triangular* and solve the system by backward substitution.

The process will come to a halt if a 0 appears in a pivot position. This can be worked around if the system is in fact consistent.

Zero pivot

Curable zero pivot

To solve the system

$$\begin{array}{cccc} x & +y & +z & = & 1 \\ 2x & +2y & +5z & = & 0 \\ 4x & +6y & +8z & = & -1\,, \end{array}$$

in matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 6 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

we work with the augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 2 & 2 & 5 & : & 0 \\ 4 & 6 & 8 & : & -1 \end{bmatrix}.$$

Here

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 5 & 0 \\ 4 & 6 & 8 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & -2 \\ 0 & 2 & 4 & -5 \end{bmatrix}.$$

We can $R_2 \longleftrightarrow R_3$, continue to

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & -5 \\ 0 & 0 & 3 & -2 \end{bmatrix},$$

and solve by backward substitution.

$\left[z = -\frac{2}{3},\ y = -\frac{7}{6},\ x = \frac{17}{6}\,.\right]$

Incurable zero pivot

If we change the coefficient of y in the third equation from 6 to 4 we find

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 5 & 0 \\ 4 & 4 & 8 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 4 & -5 \end{bmatrix}.$$

The matrix of coefficients $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 4 & 8 \end{bmatrix}$ is *singular*, in fact $\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$,

and the last two equations are *inconsistent*.

On the other hand, the system with augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 2 & 2 & 5 & : & -1 \\ 4 & 4 & 8 & : & 0 \end{bmatrix}$$

reduces to

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which is consistent, with infinitely many solutions —

$z = -1$, y is arbitrary, $x = 2 - y$.

LU decomposition

Consider Gaussian elimination for the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -2 & 0 \\ -2 & 7 & 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 5 \\ -2 \\ 8 \end{bmatrix}.$$

Here

$$\begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -2 & 0 & -2 \\ -2 & 7 & 2 & 8 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -4 & -2 & -12 \\ 0 & 8 & 3 & 13 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -4 & -2 & -12 \\ 0 & 0 & -1 & -11 \end{bmatrix}. \quad \begin{matrix} \\ \\ R_3 \to R_3 + 2R_2 \end{matrix}$$

This is equivalent to $U\mathbf{x} = \mathbf{c}$, where

$$U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -4 & -2 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 5 \\ -12 \\ -11 \end{bmatrix}.$$

[The solution can be read off by backward substitution: first $x_3 = 11$; then $-4x_2 = -12 + 2x_3 = 10$, so $x_2 = -\frac{5}{2}$; then $2x_1 = 5 - x_2 - x_3 = -\frac{7}{2}$, so $x_1 = -\frac{7}{4}$.]

This process can also be described in terms of the elementary matrices corresponding to the *ero*s that were used. Here we have

$$U = E_3E_2E_1A, \qquad \mathbf{c} = E_3E_2E_1\mathbf{b},$$

where

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \text{and } E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}.$$

Now

$$A = E_1^{-1} E_2^{-1} E_3^{-1} U.$$

Further, E_1 corresponds to $R_2 \to R_2 - 2R_1$, so E_1^{-1} corresponds to $R_2 \to R_2 + 2R_1$: so

$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Similarly, $E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ and $E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$.

Thus

$$E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} = L.$$

We have a decomposition of A in the form

$$A = LU$$

where L is *lower triangular*, with 1s in the diagonal, and U is *upper triangular*.

Clearing columns

Suppose we have an $n \times m$ matrix A [not necessarily square] with $a_{11} \neq 0$. To reduce it by row operations we apply the *ero*s $E_{R_2 \to R_2 + \lambda_2 R_1}$, $E_{R_3 \to R_3 + \lambda_3 R_1}$, $\ldots$, $E_{R_n \to R_n + \lambda_n R_1}$ where $\lambda_j = -a_{j1}/a_{11}$ $(2 \leq j \leq n)$. Their product [in any order — they commute with each other] is Q_1, say:

$$
\begin{aligned}
Q_1 &= E_{R_n \to R_n + \lambda_n R_1} \ldots E_{R_3 \to R_3 + \lambda_3 R_1} E_{R_2 \to R_2 + \lambda_2 R_1} \\
&= \begin{bmatrix} 1 & 0 & & \ldots & & 0 \\ \lambda_2 & 1 & & & & \\ \lambda_3 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & & & & & \\ \lambda_n & 0 & & \ldots & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Then $Q_1 A$ is a matrix whose first column has the form $[a_{11}, 0, \ldots, 0]^{\mathrm{T}}$:

$$
Q_1 A = A^{(1)} = \begin{bmatrix} a_{11} & \ldots & a_{1m} \\ & & \\ 0 & \ldots & a^{(1)}_{2m} \\ & & \\ \vdots & & \\ & & \\ 0 & \ldots & a^{(1)}_{nm} \end{bmatrix}
$$

Hence

$$
\begin{aligned}
A &= Q_1^{-1} A^{(1)} \\
&= \begin{bmatrix} 1 & 0 & & \ldots & & 0 \\ -\lambda_2 & 1 & & & & \\ -\lambda_3 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & & & & & \\ -\lambda_n & 0 & & \ldots & 0 & 1 \end{bmatrix} A_{(1)}
\end{aligned}
$$

Once the first column has been cleared one proceeds to the second ... The same pattern emerges. This leads to the

Quick way to find L

For each row operation of the form $R_j \to R_j + \lambda R_k$ record the number $-\lambda$, the *negative of the coefficient*, as follows:

In the above example

$$
\begin{aligned}
A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -2 & 0 \\ -2 & 7 & 2 \end{bmatrix}
&\sim \begin{array}{r} \\ 2 \\ -1 \end{array} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -4 & -2 \\ 0 & 8 & 3 \end{bmatrix} \begin{array}{l} \\ R_2 \to R_2 - (2)R_1 \\ R_3 \to R_3 - (-1)R_1 \end{array} \\
&\sim \begin{array}{r} \\ \\ -2 \end{array} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -4 & -2 \\ 0 & 0 & -1 \end{bmatrix} \begin{array}{l} \\ \\ R_3 \to R_3 - (-2)R_2 \end{array}
\end{aligned}
$$

Then form L by 'packing' these *reversed* (or *negatived) coefficients* into the format

$$
\begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix},
$$

thus obtaining

$$
L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix}.
$$

With this notation

$$
Q_1 = E_2 E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad \& \qquad Q_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}
$$

while

$$
Q_2 = E_3 \qquad \& \qquad Q_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}
$$

so that

$$
Q_1^{-1} Q_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} = L
$$

E Use the LU decomposition to solve the above system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -2 & 0 \\ -2 & 7 & 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 5 \\ -2 \\ 8 \end{bmatrix}.$$

Here $A = LU$ with

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -4 & -2 \\ 0 & 0 & -1 \end{bmatrix}.$$

We therefore need to solve the system

$$L(U\mathbf{x}) = \mathbf{b}\,.$$

First we write $\mathbf{z} = U\mathbf{x}$ and solve the system

$$L\mathbf{z} = \mathbf{b}$$

by *forward substitution* — getting $z_1 = 5$, $z_2 = -2 - 2z_1 = -12$, and then $z_3 = 8 + z_1 + 2z_2 = -11$.

Next we solve $U\mathbf{x} = \mathbf{z}$ by *backward substitution* to get

$x_3 = 11$; then $-4x_2 = -12 + 2x_3 = 10$, so $x_2 = -\frac{5}{2}$; then $2x_1 = 5 - x_2 - x_3 = -\frac{7}{2}$, so $x_1 = -\frac{7}{4}$.

Loss of significance & partial pivoting

Suppose we need to solve the system

$$\begin{aligned} 0.0001x + y &= 1 \\ x + y &= 2 \end{aligned}$$

If we use Gaussian elimination and work to *three significant figures* we get

$$\begin{bmatrix} 0.0001 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 0.0001 & 1 & 1 \\ 0 & -10\,000 & -10\,000 \end{bmatrix}$$

and then backward substitution gives $y = 1$ and $x = 0$.

On the other hand, if we reorder the equations, we get

$$\begin{bmatrix} 1 & 1 & 2 \\ 0.0001 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

and then backward substitution gives $y = 1$ and $x = 1$.

The exact solution is $x = \dfrac{10^4}{10^4 - 1} = 1.\dot{0}00\dot{1} = 1.0001\,0001\,\ldots$ and $y = \dfrac{10^4 - 2}{10^4 - 1} = 0.\dot{9}99\dot{8} = 0.9998\,9998\ldots$. The second approximation is much closer than the first.

So we see that if a pivot is very small we may lose significance. We should therefore aim to choose pivots as large [in *modulus*] as possible.

We shall mainly employ this procedure of *partial pivoting*.

[*Full pivoting* entails inspecting the columns and performing column operations too, in order to maximise the pivots. The extra complications and cost usually outweigh the benefits, so we shall not examine this process.]

E Use Gaussian elimination with partial pivoting to solve the system

$$
\begin{array}{rrrcl}
x & +2y & +3z & = & 2 \\
x & -2y & +z & = & 1 \\
-2x & +y & -z & = & 0.
\end{array}
$$

Here

$$
\begin{aligned}
\begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & -2 & 1 & 1 \\ -2 & 1 & -1 & 0 \end{bmatrix}
&\sim \begin{bmatrix} -2 & 1 & -1 & 0 \\ 1 & -2 & 1 & 1 \\ 1 & 2 & 3 & 2 \end{bmatrix} \quad R_1 \longleftrightarrow R_3 \\
&\sim \begin{bmatrix} -2 & 1 & -1 & 0 \\ 0 & -\frac{3}{2} & \frac{1}{2} & 1 \\ 0 & \frac{5}{2} & \frac{5}{2} & 2 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + \frac{1}{2}R_1 \\ R_3 \to R_3 + \frac{1}{2}R_1 \end{matrix} \\
&\sim \begin{bmatrix} -2 & 1 & -1 & 0 \\ 0 & \frac{5}{2} & \frac{5}{2} & 2 \\ 0 & -\frac{3}{2} & \frac{1}{2} & 1 \end{bmatrix} \quad R_2 \longleftrightarrow R_3 \\
&\sim \begin{bmatrix} -2 & 1 & -1 & 0 \\ 0 & \frac{5}{2} & \frac{5}{2} & 2 \\ 0 & 0 & 2 & \frac{11}{5} \end{bmatrix}. \quad R_3 \to R_3 + \frac{3}{5}R_1
\end{aligned}
$$

Back substitution yields $z = \dfrac{11}{10}$, then $y = -\dfrac{3}{10}$, and then $x = -\dfrac{7}{10}$.

LU-pivoting

Consider the following example.

E

$$
\begin{aligned}
A &= \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 0 \\ 4 & 0 & 4 \end{bmatrix} \\
&\sim \begin{bmatrix} 4 & 0 & 4 \\ 2 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix} \quad R_1 \longleftrightarrow R_3 \\
&\sim \begin{array}{c} \\ \frac{1}{2} \\ \frac{1}{4} \end{array} \begin{bmatrix} 4 & 0 & 4 \\ 0 & 2 & -2 \\ 0 & 3 & 0 \end{bmatrix} \\
&\sim \begin{bmatrix} 4 & 0 & 4 \\ 0 & 3 & 0 \\ 0 & 2 & -2 \end{bmatrix} \quad R_2 \longleftrightarrow R_3 \\
&\sim \begin{array}{c} \\ \\ \frac{2}{3} \end{array} \begin{bmatrix} 4 & 0 & 4 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} = U.
\end{aligned}
$$

Here there are two interchanges.

The first is described by $P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, and the second by $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$:
the total (interchange) effect is $P = P_2 P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Now let us start again, from PA. We find that

$$
\begin{aligned}
PA &= \begin{array}{c} \\ \\ \\ \end{array} \begin{bmatrix} 4 & 0 & 4 \\ 1 & 3 & 1 \\ 2 & 2 & 0 \end{bmatrix} \\
&\sim \begin{array}{c} \\ \frac{1}{4} \\ \frac{1}{2} \end{array} \begin{bmatrix} 4 & 0 & 4 \\ 0 & 3 & 0 \\ 0 & 2 & -2 \end{bmatrix} \\
&\sim \begin{array}{c} \\ \\ \frac{2}{3} \end{array} \begin{bmatrix} 4 & 0 & 4 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}
\end{aligned}
$$

So we see that $PA = LU$, where U is as above and $L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & \frac{2}{3} & 1 \end{bmatrix}$.

Note that the *reversed coefficients* that appeared in clearing the first column of A appear again when clearing the first column of PA: and again in the *opposite* order. This is *systematic*. In this example

$$
P_1 A = \begin{bmatrix} 4 & 0 & 4 \\ 2 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}
$$

so

$$
\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{4} & 0 & 1 \end{bmatrix} P_1 A = \begin{bmatrix} 4 & 0 & 4 \\ 0 & 2 & -2 \\ 0 & 3 & 0 \end{bmatrix}
$$

Next

$$
P_2 \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{4} & 0 & 1 \end{bmatrix} P_1 A = \begin{bmatrix} 4 & 0 & 4 \\ 0 & 3 & 0 \\ 0 & 2 & -2 \end{bmatrix}
$$

so

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} P_2 \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{4} & 0 & 1 \end{bmatrix} P_1 A = \begin{bmatrix} 4 & 0 & 4 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} = U\,.
$$

It follows that

$$
\begin{aligned}
P_1 A &= \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{4} & 0 & 1 \end{bmatrix}^{-1} P_2^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix}^{-1} U \\
&= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & 0 & 1 \end{bmatrix} P_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix} U
\end{aligned}
$$

and therefore

$$
\begin{aligned}
PA = P_2 P_1 A &= \left\{ P_2 \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & 0 & 1 \end{bmatrix} P_2 \right\} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix} U \\
&= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix} U \\
&= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & \frac{2}{3} & 1 \end{bmatrix} U \\
&= LU\,.
\end{aligned}
$$

E Use the LU decomposition *with partial pivoting* to solve the system of equations

$$
\begin{array}{rrrcl}
x & +y & & = & 2 \\
2x & & -z & = & 1 \\
2x & +3y & -z & = & 3.
\end{array}
$$

Here

$$
\begin{aligned}
A &= \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 2 & 3 & -1 \end{bmatrix} \\
&\sim \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \\ 2 & 3 & -1 \end{bmatrix} \quad R_1 \longleftrightarrow R_2
\end{aligned}
$$

$$
\begin{aligned}
&\sim \quad \begin{array}{c} \\ \frac{1}{2} \\ 1 \end{array} \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 3 & 0 \end{bmatrix} \\
&\sim \quad \phantom{\frac{1}{2}} \begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 1 & \frac{1}{2} \end{bmatrix} \quad R_2 \longleftrightarrow R_3 \\
&\sim \quad \begin{array}{c} \\ \\ \frac{1}{3} \end{array} \begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = U.
\end{aligned}
$$

Here

$$
L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{1}{2} & \frac{1}{3} & 1 \end{bmatrix}
$$

and

$$
P = E_{R_2 \longleftrightarrow R_3} E_{R_1 \longleftrightarrow R_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.
$$

So

$$
PA = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{1}{2} & \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}.
$$

Note again that the *reversed coefficients* that appeared in clearing the first column of A appear again when clearing the first column of PA: but in the *opposite* order.

The system is thus equivalent to

$$
PA \begin{bmatrix} x \\ y \\ z \end{bmatrix} = P \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.
$$

First we solve

$$
L \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.
$$

Forward substitution immediately gives $u = 1$, then $v = 3 - u = 2$, and then $w = 2 - \frac{1}{2}u - \frac{1}{3}v = \frac{5}{6}$.

Now we solve

$$U \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ \frac{5}{6} \end{bmatrix}$$

by backward substitution, to find that $z = \frac{5}{3}$, then $y = \frac{2}{3}$, and then $x = (1+z)/2 = \frac{4}{3}$.

E Use the *LU* decomposition *with partial pivoting* to solve the system of equations

$$\begin{array}{rrrrcr} x & +4y & +3z & -t & = & 6 \\ -3x & & +2z & & = & -3 \\ 4x & +3y & & +2t & = & 5 \\ 2x & -y & +3z & -5t & = & 6. \end{array}$$

Here

$$\begin{aligned}
A &= \begin{bmatrix} 1 & 4 & 3 & -1 \\ -3 & 0 & 2 & 0 \\ 4 & 3 & 0 & 2 \\ 2 & -1 & 3 & -5 \end{bmatrix} \\
&\sim \begin{bmatrix} 4 & 3 & 0 & 2 \\ -3 & 0 & 2 & 0 \\ 1 & 4 & 3 & -1 \\ 2 & -1 & 3 & -5 \end{bmatrix} \quad R_1 \longleftrightarrow R_3 \\
&\sim \begin{array}{c} \\ -\frac{3}{4} \\ \frac{1}{4} \\ \frac{1}{2} \end{array} \begin{bmatrix} 4 & 3 & 0 & 2 \\ 0 & \frac{9}{4} & 2 & \frac{3}{2} \\ 0 & \frac{13}{4} & 3 & -\frac{3}{2} \\ 0 & -\frac{5}{2} & 3 & -6 \end{bmatrix} \\
&\sim \begin{bmatrix} 4 & 3 & 0 & 2 \\ 0 & \frac{13}{4} & 3 & -\frac{3}{2} \\ 0 & \frac{9}{4} & 2 & \frac{3}{2} \\ 0 & -\frac{5}{2} & 3 & -6 \end{bmatrix} \quad R_2 \longleftrightarrow R_3
\end{aligned}$$

$$
\begin{aligned}
&\sim \quad \begin{array}{c} \\ \\ \frac{9}{13} \\ -\frac{10}{13} \end{array} \begin{bmatrix} 4 & 3 & 0 & 2 \\ 0 & \frac{13}{4} & 3 & -\frac{3}{2} \\ 0 & 0 & -\frac{1}{13} & \frac{33}{13} \\ 0 & 0 & \frac{69}{13} & -\frac{93}{13} \end{bmatrix} \\
&\sim \quad \begin{bmatrix} 4 & 3 & 0 & 2 \\ 0 & \frac{13}{4} & 3 & -\frac{3}{2} \\ 0 & 0 & \frac{69}{13} & -\frac{93}{13} \\ 0 & 0 & -\frac{1}{13} & \frac{33}{13} \end{bmatrix} \quad R_3 \longleftrightarrow R_4 \\
&\sim \quad \begin{array}{c} \\ \\ \\ -\frac{1}{69} \end{array} \begin{bmatrix} 4 & 3 & 0 & 2 \\ 0 & \frac{13}{4} & 3 & -\frac{3}{2} \\ 0 & 0 & \frac{69}{13} & -\frac{93}{13} \\ 0 & 0 & 0 & \frac{56}{23} \end{bmatrix} \qquad = \; U
\end{aligned}
$$

Here $PA = LU$ where P is determined by the row operations done —

$$
P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}
$$

and L is found by applying the proper modifications to the matrix formed using the left-hand annotations —

$$
\begin{aligned}
& \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ -\frac{3}{4} & 1 & \cdot & \cdot \\ \frac{1}{4} & \frac{9}{13} & 1 & \cdot \\ \frac{1}{2} & -\frac{10}{13} & -\frac{1}{69} & 1 \end{bmatrix} \\
\longrightarrow \quad & \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \frac{1}{4} & 1 & \cdot & \cdot \\ -\frac{3}{4} & \frac{9}{13} & 1 & \cdot \\ \frac{1}{2} & -\frac{10}{13} & -\frac{1}{69} & 1 \end{bmatrix} \quad R_2 \longleftrightarrow R_3 \quad [\textit{first column}] \\
\longrightarrow \quad & \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \frac{1}{4} & 1 & \cdot & \cdot \\ \frac{1}{2} & -\frac{10}{13} & 1 & \cdot \\ -\frac{3}{4} & \frac{9}{13} & -\frac{1}{69} & 1 \end{bmatrix} \quad R_3 \longleftrightarrow R_4 \quad [\textit{first \& second columns}] \\
= \quad & L\,.
\end{aligned}
$$

The original system

$$AX = B$$

is equivalent to

$$PA\,X = P\,B$$

ie to

$$LU\,X = P\,B$$

so we first solve

$$LW = PB \quad \text{where} \quad W = \begin{bmatrix} r \\ s \\ t \\ u \end{bmatrix}$$

ie

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \frac{1}{4} & 1 & \cdot & \cdot \\ \frac{1}{2} & -\frac{10}{13} & 1 & \cdot \\ -\frac{3}{4} & \frac{9}{13} & -\frac{1}{69} & 1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \\ u \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 6 \\ -3 \end{bmatrix}$$

to find that $r = 5$, $s = 6 - \frac{r}{4} = \frac{19}{4}$, $t = 6 - \frac{r}{2} + \frac{10}{13}s = 6 - \frac{5}{2} + \frac{10}{13}\frac{19}{4} = \frac{93}{13}$, $u = -3 + \frac{1}{69}t - \frac{9}{13}s + \frac{3}{4}r = \ldots = -\frac{56}{23}$.

Next we solve

$$UX = W$$

ie

$$\begin{bmatrix} 4 & 3 & 0 & 2 \\ 0 & \frac{13}{4} & 3 & -\frac{3}{2} \\ 0 & 0 & \frac{69}{13} & -\frac{93}{13} \\ 0 & 0 & 0 & \frac{56}{23} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ \frac{19}{4} \\ \frac{93}{13} \\ -\frac{56}{23} \end{bmatrix}$$

and find that $t = -1$, $\frac{69}{13}z + \frac{93}{13} = \frac{93}{13}$: so $z = 0$: then $\frac{13}{4}y - \frac{3}{2}(-1) = \frac{19}{4}$, from which $y = 1$: then $4x + 3 - 2 = 5$, from which $x = 1$.

LDU-decomposition

If we decompose A as LU then the diagonal entries of L are all equal to 1, but those of U need not be.

We can, however, define an invertible *diagonal* matrix D by putting $d_{jj} = u_{jj}$ $(1 \leq j \leq n)$ if $u_{jj} \neq 0$, and $d_{jj} = 1$ if $u_{jj} = 0$.

Then V, defined as $D^{-1}U$, is also upper triangular. All its *nonzero* diagonal entries are equal to 1, and $A = L\,D\,V$.

In the last-but-one example

$$\begin{aligned}
PA = LU \quad &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{1}{2} & \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{1}{2} & \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = LDV.
\end{aligned}$$

1.5 Number of operations

The number of multiplications/divisions required to effect a row operation of the form $R_j \to R_j + \lambda R_k$ is, approximately, n. To clear the first column of an $n \times n$ matrix therefore requires $\approx n^2$ such operations.

To reduce an $n \times n$ matrix to upper diagonal form thus takes roughly

$$n^2 + (n-1)^2 + \ldots + 1 \approx \frac{n^3}{3}$$

multiplications.