Mathematics 1T (Algebra)

Summary of Week #9

- **<u>The Binomial Theorem</u>**   For any $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r \; . \tag{1}$$

The expression that appears on the RHS is called the *binomial expansion.*

OBS.

1. Note that the right-hand side in (1) stands for

$$\sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a b^{n-1} + \binom{n}{n} b^n,$$

and the coefficients that appear in this sum are precisely those in row n of Pascal's Triangle.

2. If we replace b by $-b$ in (1), we obtain:

$$(a-b)^n = \sum_{r=0}^{n} \binom{n}{r} (-1)^r a^{n-r} b^r \; , \tag{2}$$

where

$$\sum_{r=0}^{n} \binom{n}{r} (-1)^r a^{n-r} b^r = \binom{n}{0} a^n - \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 - \dots$$
$$+ \binom{n}{n-1} (-1)^{n-1} a b^{n-1} + \binom{n}{n} (-1)^n b^n \; .$$

Typical problems related to the *Binomial Theorem* involve finding the coefficient of some power of x in the binomial expansion of an expression of the form $(a+b)^n$, where n is usually given and $a \equiv a(x)$, $b \equiv b(x)$ are also given expressions that depend on x. You should study carefully the following example.

Example #1:

Find the coefficient of x^4 in the expansion of

$$\left(2x^3 - \frac{1}{x^2} \right)^8 \; . \tag{3}$$

Solution

Use (1) with $a = 2x^3$, $b = -1/x^2$, and $n = 8$. The *typical* term in the binomial expansion associated with (3) is

$$\binom{8}{r}(2x^3)^{8-r}\left(-\frac{1}{x^2}\right)^r = \binom{8}{r}2^{8-r}(-1)^r x^{24-5r},$$

where $r = 0, 1, 2, \ldots, 8$. We have separated the constant parts of the factor from the power of x, and combined the latter into a single power of x.

For the term in x^4 we must have

$$24 - 5r = 4, \qquad \text{i.e.} \qquad r = 4.$$

Thus, the required coefficient is

$$\binom{8}{4}2^4(-1)^4 = 1120.$$

<u>Related issues:</u> *Are there any terms which do not contain x in the binomial expansion of (3)? How about terms in x^{14}?*

To find the term which does not contain x, we notice that such a possible term would correspond to

$$24 - 5r = 0, \qquad \text{i.e.} \qquad r = \frac{24}{5}.$$

However, this solution is not acceptable as r must be a positive integer between 0 and 8. Hence we conclude that there are no free terms in (3).

The other question can be tackled in a similar way. The term in x^{14} requires

$$24 - 5r = 14, \qquad \text{i.e.} \qquad r = 2, \quad \text{etc.}$$

- **Applications of the Binomial Theorem**   There are a number of occasions where it is necessary to re-write

$$\cos(n\theta) \qquad \text{and} \qquad \sin(n\theta)$$

in terms of powers of

$$\cos(\theta) \qquad \text{and} \qquad \sin(\theta),$$

or vice-versa. This can be achieved relatively easily with the help of the *Binomial Theorem*.

1. **Multiple angles to powers: the c+is method**

For any $\theta \in \mathbb{R}$, we define the *complex exponential*

$$e^{i\theta} \equiv \cos(\theta) + i\sin(\theta), \tag{4}$$

where $i = \sqrt{-1}$ is the imaginary unit.

De Moivre's Theorem states that

$$\left(e^{i\theta}\right)^n = e^{in\theta} \qquad \text{for all} \quad n \in \mathbb{N}.$$

According to (4), this can also be expressed as

$$(\cos(\theta) + \mathrm{i}\sin(\theta))^n = \cos(n\theta) + \mathrm{i}\sin(n\theta). \tag{5}$$

In what follows, we shall write $\cos(\theta)$ and $\sin(\theta)$ as c and s, respectively. With this notation in hand, we can now re-write (5) as

$$\boxed{\cos(n\theta) + \mathrm{i}\sin(n\theta) = (c + \mathrm{i}s)^n}. \tag{6}$$

Obtaining $\cos(n\theta)$ in terms of powers of c and s involves a sequence of simple steps:

(a) $\boxed{\textbf{STEP 1}}$ Expand the RHS of (6) with the help of the *Binomial Theorem*. The result will be a complex expression, i.e. something of the form

$$(\ldots) + \mathrm{i}(\ldots), \tag{7}$$

where the dots stand for sums involving various powers of c and s.

(b) $\boxed{\textbf{STEP 2}}$ Equate the real part of the LHS of (6) and the real part of (7). If $\sin(n\theta)$ is required in terms of powers of c and s, then equate the imaginary parts of the expressions involved in the previous case.

$\boxed{\textbf{OBS.}}$ In both cases above, the final results obtained will be in terms of powers of both c and s. To get an answer that involves only c (or only s), use

$$\sin^2(\theta) + \cos^2(\theta) = 1,$$

which shows that

$$\boxed{c^2 = 1 - s^2 \qquad \text{and} \qquad s^2 = 1 - c^2}.$$

Example #2:

Express $\cos(5\theta)$ as a polynomial in $\cos(\theta)$.

Solution

$$\begin{aligned}
\cos(5\theta) + \mathrm{i}\sin(5\theta) &= (c + \mathrm{i}s)^5 \\
&= c^5 + 5\,c^4(\mathrm{i}s) + 10\,c^3(\mathrm{i}s)^2 + 10\,c^2(\mathrm{i}s)^3 + 5\,c(\mathrm{i}s)^4 + (\mathrm{i}s)^5 \\
&= c^5 - 10\,c^3s^2 + 5\,cs^4 + \mathrm{i}\left(5\,c^4s - 10\,c^2s^3 + s^5\right).
\end{aligned}$$

Equating real parts, we obtain

$$\cos(5\theta) = c^5 - 10\,c^3s^2 + 5\,cs^4. \tag{8}$$

Note that the RHS of (8) depends on both c and s, but we need an expression that involves only c. To this end, remember that

$$s^2 = 1 - c^2, \tag{9}$$

and putting together (9) and (8), we find

$$\begin{aligned}
\cos(5\theta) &= c^5 - 10\,c^3(1-c^2) + 5\,c(1-c^2)^2 \\
&= 16\,c^5 - 20\,c^3 + 5\,c \\
&= 16\cos^5(\theta) - 20\cos^3(\theta) + 5\cos(\theta)\,.
\end{aligned}$$

2. **Powers to multiple angles: the $z + \frac{1}{z}$ method**

   If
   $$z = \cos(\theta) + i\sin(\theta)\,,$$

   then

   $$z + \frac{1}{z} = 2\cos(\theta)\,, \tag{10a}$$

   $$z - \frac{1}{z} = 2i\sin(\theta) \tag{10b}$$

   and, for any $n \in \mathbb{N}$,

   $$z^n + \frac{1}{z^n} = 2\cos(n\theta)\,, \tag{11a}$$

   $$z^n - \frac{1}{z^n} = 2i\sin(n\theta)\,. \tag{11b}$$

   To express powers of $\cos(\theta)$ and $\sin(\theta)$ in terms of multiple angles, first express corresponding powers of $2\cos(\theta)$ and $2i\sin(\theta)$ in terms of z and $1/z$ using (10a) or (10b), and then use the *Binomial Theorem* in conjunction with (11a) or (11b), depending on the case.

   Example #3:

   Express $\cos^6(\theta)$ in the form
   $$a\cos(6\theta) + b\cos(4\theta) + c\cos(2\theta) + d\,,$$

   where a, b, c, d are constants that need to be determined.

   Solution

   Starting with (10a), we obtain successively

   $$\begin{aligned}
   (2\cos(\theta))^6 &= \left(z + \frac{1}{z}\right)^6 \\
   &= z^6 + 6\,z^4 + 15\,z^2 + 20 + 15\,\frac{1}{z^2} + 6\,\frac{1}{z^4} + \frac{1}{z^6} \\
   &= \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20\,. \tag{12}
   \end{aligned}$$

   On the other hand, according to (11a) the following are true,

   $$z^6 + \frac{1}{z^6} = 2\cos(6\theta), \quad z^4 + \frac{1}{z^4} = 2\cos(4\theta), \quad\text{and}\quad z^2 + \frac{1}{z^2} = 2\cos(2\theta)\,. \tag{13}$$

   Finally, the required result follows by using (13) in (12),

   $$\cos^6(\theta) = \frac{1}{32}\cos(6\theta) + \frac{3}{16}\cos(4\theta) + \frac{15}{32}\cos(2\theta) + \frac{5}{16}\,.$$

   Hence, $a = 1/32$, $b = 3/16$, $c = 15/32$, and $d = 5/16$.