

Mathematics 1T (Algebra)

Summary of Week #8

- **Principles of counting** There are essentially three such fundamental principles:

I Let \mathcal{A}_1 and \mathcal{A}_2 be two events. If the first event can occur in N_1 ways, and the second in N_2 ways, then

\mathcal{A}_1 followed by \mathcal{A}_2 can occur in $N_1 \times N_2$ ways.

This result can be generalised in an obvious way:

If $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_p$ are p events ($p \geq 2$) such that \mathcal{A}_1 can occur in N_1 ways, \mathcal{A}_2 can occur in N_2 ways, ..., and \mathcal{A}_p can occur in N_p ways, then

\mathcal{A}_1 followed by \mathcal{A}_2 followed by ... followed by \mathcal{A}_p

can occur in

$$N_1 \times N_2 \times \dots \times N_p \text{ ways}$$

II Let \mathcal{A}_1 and \mathcal{A}_2 be two events. If the first event can occur in N_1 ways, and the second in N_2 ways, then

\mathcal{A}_1 or \mathcal{A}_2 can occur in $N_1 + N_2$ ways.

Again, this principle can be extended to accommodate an arbitrary (but finite) number of events, as shown above. For example, if we have three events, $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$, and

\mathcal{A}_1 can occur in N_1 ways,

\mathcal{A}_2 can occur in N_2 ways,

\mathcal{A}_3 can occur in N_3 ways,

then

\mathcal{A}_1 or \mathcal{A}_2 or \mathcal{A}_3 can occur in $N_1 + N_2 + N_3$ ways.

III Assume that some event \mathcal{A} , which can occur in N ways, can be broken down into several sub-events, $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_p$. By that we mean that \mathcal{A} is the event

\mathcal{A}_1 or \mathcal{A}_2 or ... or \mathcal{A}_p .

Assume further that \mathcal{A}_1 can occur in N_1 ways, \mathcal{A}_2 can occur in N_2 ways, ..., and \mathcal{A}_p can occur in N_p ways. Then the number of ways in which \mathcal{A} can occur *without* \mathcal{A}_j taking place (for some $1 \leq j \leq p$) is given by

$$N - N_j.$$

Example #1:

In a certain city, there are four bus companies: A , B , C , D . Between the city centre and a suburb, A runs 6 bus routes, B runs 4, C runs 3, and D runs 1.

1. If I decide to travel from the centre to the suburb using an A or a C bus, how many choices do I have?
2. If I decide to travel from the city centre to the suburb *not* using an A bus, how many routes to chose from do I have?
3. On another occasion I decide to travel to the suburb by an A bus, and return back to the city centre using again an A bus, but taking a different route. What are my possibilities?

Solution

1). The solution for this question follows immediately by applying II. Note that we can define the following events

$$\begin{aligned} \mathcal{A}_1 &= \text{travelling from the centre to the suburb on an } A \text{ bus} & (N_1 = 6), \\ \mathcal{A}_2 &= \text{travelling from the centre to the suburb on a } B \text{ bus} & (N_2 = 4), \\ \mathcal{A}_3 &= \text{travelling from the centre to the suburb on a } C \text{ bus} & (N_3 = 3), \\ \mathcal{A}_4 &= \text{travelling from the centre to the suburb on a } D \text{ bus} & (N_4 = 1), \end{aligned}$$

Travelling from the centre to the suburb using either an A or a C bus is the event

$$\mathcal{A}_1 \text{ or } \mathcal{A}_3,$$

so the number of ways in which this event can occur is equal to $N_1 + N_3 = 9$.

2). The second question is even easier. If we think of “*travelling from the centre to the suburb*” as being the event

$$\mathcal{A}_1 \text{ or } \mathcal{A}_2 \text{ or } \mathcal{A}_3 \text{ or } \mathcal{A}_4,$$

then the number of ways in which this event can occur is equal to 14, that is $N = 14$ in III. Thus, the number of possibilities in which I can travel to the suburb *not* using an A bus is equal to $N - N_1 = 8$.

3). The last question is a typical situation that involves applying I. To this end, we consider the following new events,

$$\begin{aligned} \mathcal{A}_1 &= \text{travelling from the centre to the suburb on an } A \text{ bus} & (N_1 = 6), \\ \mathcal{A}_2 &= \text{travelling from the suburb to the city centre on an } A \text{ bus,} \\ &\quad \text{excluding the bus used on the outward journey} & (N_2 = 5), \end{aligned}$$

We are required to find the number of ways in which the event

$$\mathcal{A}_1 \text{ followed by } \mathcal{A}_2$$

can occur, and that is going to be equal to $N_1 \times N_2 = 30$.

• Permutations & combinations

1. A **permutation** of k objects from the n objects means a *list* (set) of k of the objects, *all different and in a definite order*.

2. A **combination** of k of the n objects means a *list* (set) consisting of k of the objects, *the order in which the k objects are chosen being regarded as irrelevant or immaterial*.

3. **Notation:**

(a) The number of *permutations* of k objects taken from a given set with n elements is denoted by

$${}^n P_k .$$

(b) The number of *combinations* of k objects taken from a given set with n elements is denoted by

$$\binom{n}{k} \quad \text{or} \quad {}^n C_k .$$

4. **Important formulae:**

(a) The following formulae must be memorised:

$${}^n P_k = n(n-1)(n-2)\dots(n-k+1) \tag{1a}$$

$${}^n P_k = \frac{n!}{(n-k)!} \tag{1b}$$

$$\binom{n}{k} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} \tag{1c}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} . \tag{1d}$$

Also,

$$\binom{n}{k} = \binom{n}{n-k} \tag{2a}$$

$$\binom{n}{0} = \binom{n}{n} = 1 \quad \text{and} \quad \binom{n}{1} = \binom{n}{n-1} = n \tag{2b}$$

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} . \tag{2c}$$

Example #2:

A club has 20 members: 8 men and 12 women. A committee of 5 members is to be chosen. In how many ways can this be done if the committee is to contain.

1. 2 men and 3 women?
2. *at least* 3 women?
3. *at least* 1 woman?

Solution

1). The counting principle **I** applies here. A committee of 5 members can be assembled by choosing 2 men *and then* choosing 3 women (if the women are chosen first, nothing changes). Consider the events

$$\begin{aligned}\mathcal{A}_1 &= \text{choosing 2 men from 8,} & N_1 &= \binom{8}{2}, \\ \mathcal{A}_2 &= \text{choosing 3 women from 12,} & N_2 &= \binom{12}{3}.\end{aligned}$$

We must recognise that the first question is about the number of ways in which the event

\mathcal{A}_1 followed by \mathcal{A}_2

can occur. According to **I**, this number is equal to $N_1 \times N_2$, i.e.

$$\binom{8}{2} \times \binom{12}{3} = 6,160.$$

2). The main idea is to break down “at least 3 women” into precise, non-overlapping cases that are easily counted. Consider the following events:

$$\begin{aligned}\mathcal{A}_1 &= \text{assembling a committee with exactly 3 women (and 2 men),} & N_1 &= \binom{12}{3} \times \binom{5}{2}, \\ \mathcal{A}_2 &= \text{assembling a committee with exactly 4 women (and 1 man),} & N_2 &= \binom{12}{4} \times \binom{5}{1}, \\ \mathcal{A}_3 &= \text{assembling a committee with exactly 5 women (and 0 men),} & N_3 &= \binom{12}{5} \times \binom{5}{0}.\end{aligned}$$

We are asked to find the number of ways in which the event

\mathcal{A}_1 or \mathcal{A}_2 or \mathcal{A}_3

can occur. By **II** this number is $N_1 + N_2 + N_3$, i.e.

$$\binom{12}{3} \times \binom{5}{2} + \binom{12}{4} \times \binom{5}{1} + \binom{12}{5} \times \binom{5}{0} = 10,912.$$

3). This question can be answered exactly the same as the previous one, that is by adding up the number of ways in which we can assemble committees with 1, 2, 3, 4 and, respectively, 5 women. However, this is not the simplest way to get the answer.

If we let \mathcal{A} be the event,

$$\mathcal{A} = \text{assembling a committee of 5 people,} \quad N = \binom{20}{5},$$

the next thing to notice is that this can be broken down into 6 sub-events corresponding to selecting committees with 1, 2, 3, 4, 5, 6 and, respectively, 0 women. Let's call this last event \mathcal{A}_0 , i.e.

$$\mathcal{A}_0 = \text{assembling a committee containing 0 women (and 5 men),} \quad N_0 = \binom{12}{0} \times \binom{8}{5}.$$

Finally, the required answer is obtained by applying $\boxed{\text{III}}$, and we find that the number of ways of choosing a committee with at least 1 woman is equal to

$$N - N_0 = \binom{20}{5} - \binom{12}{0} \times \binom{8}{5} = 15,448.$$

• **Pascal's Triangle** represents a triangular array constructed as follows (see below):

- The first row has only two elements, both equal to 1.
- All the other rows are built by:
 - * placing 1 at the beginning and the end of each row.
 - * every other entry is the sum of the nearest two entries in the row above (i.e. the one above it and to the right, and the one above it and to the left).

Row 1				1		1						
Row 2			1		2		2					
Row 3			1		3		3		1			
Row 4		1		4		6		4	1			
Row 5		1		5		10		10	5	1		
Row 6	1		6		15		20		15	6	1	
Row 7	1		7		21		35		35	21	7	1

• The numbers in row n of Pascal's Triangle are

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n-1}, \binom{n}{n}.$$