

Mathematics 1T (Algebra)

Summary of Week #7

- **Mathematical Induction** (preliminaries)

1. Sigma notation If $f : \mathbb{Z} \rightarrow \mathbb{R}$, then

$$\sum_{i=1}^n f(i) \quad \text{denotes the sum} \quad f(1) + f(2) + \cdots + f(n). \quad (1)$$

Note that the index i in (1) is a *dummy variable*. This means that its name does not make any difference in the final result. So, for example,

$$\sum_{i=1}^n f(i) = \sum_{r=1}^n f(r) = \sum_{j=1}^n f(j), \quad \text{etc.}$$

Some elementary rules for manipulating expressions involving the sigma notation are listed below

$$\alpha \left(\sum_{i=1}^n f(i) \right) = \sum_{i=1}^n (\alpha f(i)), \quad \text{for all } \alpha \in \mathbb{R}.$$

$$\sum_{i=1}^n (f(i) + g(i)) = \sum_{i=1}^n f(i) + \sum_{i=1}^n g(i), \quad \text{where } g : \mathbb{Z} \rightarrow \mathbb{R}.$$

$$\sum_{i=1}^{n+1} f(i) = \left(\sum_{i=1}^n f(i) \right) + f(n+1).$$

2. Mathematical induction is often used to deal with objects which have a *recursive* definition. A function

$$F : \mathbb{N} \rightarrow \mathbb{R}$$

is said to be **recursive** if $F(n+1)$ is defined in terms of $F(n)$.

Examples:

(a) $F(n) = n!$

$$F(n+1) = (n+1)! = n! \times (n+1) = (n+1)F(n).$$

(b) $F(n) = \sum_{i=1}^n i^2$

$$F(n+1) = \sum_{i=1}^{n+1} i^2 = \left(\sum_{i=1}^n i^2 \right) + (n+1)^2 = F(n) + (n+1)^2.$$

(c) $F(n) = A^n$ where $A \in M_{p \times q}(\mathbb{R})$

$$F(n+1) = A^{n+1} = A^n \cdot A = F(n)A.$$

(d) $F(n) =$ derivative of order n w.r.t x of a given function $Y(x)$, i.e.

$$F(n) = \frac{d^n Y}{dx^n}.$$

Note that

$$F(n+1) = \frac{d^{n+1} Y}{dx^{n+1}} = \frac{d}{dx} \left(\frac{d^n Y}{dx^n} \right).$$

• **The Principle of Mathematical Induction** *In order to prove that a statement $\mathcal{P}(n)$ is true for all $n \in \mathbb{N}$ it is sufficient to do the following two things:*

1. *Prove that $\mathcal{P}(n)$ is true for $n = 1$.*
2. *Prove that if $\mathcal{P}(n)$ is true in the case $n = k$, then $\mathcal{P}(n)$ is true in the case $n = k + 1$.*

OBS.

- we think of k as being a single, but *arbitrary* value.
- we are being asked to prove that **if** $\mathcal{P}(n)$ is true for any one value $n = k$, **then** it is also true for the **next** value, $n = k + 1$.

Example: Prove by induction that for all $n \in \mathbb{N}$,

$$\sum_{r=1}^n r^3 = \frac{1}{4} n^2 (n+1)^2. \quad (2)$$

Solution

STEP 1 Let $\mathcal{P}(n)$ denote the statement (2) above, that is,

$$\mathcal{P}(n) : \quad \sum_{r=1}^n r^3 = \frac{1}{4} n^2 (n+1)^2, \quad \text{for all } n \in \mathbb{N} \quad (3)$$

STEP 2 Verify that $\mathcal{P}(n)$ is true for $n = 1$. This involves evaluating independently both the right-hand side (RHS) and the left-hand side (LHS) of (2) for $n = 1$. The outcome of these (trivial) calculations should be an obvious equality (see below).

$$\text{LHS} = \sum_{r=1}^1 r^3 = 1^3 = 1,$$

$$\text{RHS} = \frac{1}{4} \times 1^2 \times 2^2 = \frac{4}{4} = 1,$$

so $\mathcal{P}(n)$ is true for $n = 1$.

STEP 3 Suppose now that $\mathcal{P}(n)$ is true in the case $n = k$, i.e. suppose that

$$\sum_{r=1}^k r^3 = \frac{1}{4}k^2(k+1)^2. \quad (4)$$

STEP 4 With the hypothesis introduced at STEP 3, our aim is to *deduce* that $\mathcal{P}(n)$ is also true in the case $n = k + 1$, i.e. we want to show that

$$\sum_{r=1}^{k+1} r^3 = \frac{1}{4}(k+1)^2(k+2)^2. \quad (5)$$

Broadly speaking, this involves re-writing the LHS of (5) so that we can take advantage of (4). Observe the calculations included below:

$$\begin{aligned} \text{LHS of (5)} &= \sum_{r=1}^{k+1} r^3 \\ &= \sum_{r=1}^k r^3 + (k+1)^3 \\ &= \frac{1}{4}k^2(k+1)^2 + (k+1)^3 && \text{[according to (4)]} \\ &= (k+1)^2 \left\{ \frac{1}{4}k^2 + (k+1) \right\} \\ &= \frac{1}{4}(k+1)^2(k^2 + 4k + 4) \\ &= \frac{1}{4}(k+1)^2(k+2)^2, \end{aligned}$$

and so (5) is true for $n = k + 1$.

STEP 5 Thus, if $\mathcal{P}(n)$ is true in the case $n = k$, then it is also true in the case $n = k + 1$.

STEP 6 It now follows by induction that $\mathcal{P}(n)$ is true for all positive integers n .