

Mathematics 1T (Algebra)

Summary of Week #6

- **Hypothetical statements** Many mathematical statements are of the form

$$\boxed{\text{If } \mathcal{P} \text{ then } \mathcal{Q}}, \quad (1)$$

where

\mathcal{P} = is the **hypothesis**,

\mathcal{Q} = is the **conclusion**.

Notation: $\mathcal{P} \implies \mathcal{Q}$ is read “ \mathcal{P} implies \mathcal{Q} ” and means exactly the same as (1).

Examples (of mathematical statements in the above form):

1. If $x > 4$ then $x^2 > 16$.
2. If m is an even integer then m^2 is even.
3. If A, B, C are three points such that $\overrightarrow{AB} = 2\overrightarrow{BC}$ then A, B, C are collinear.

We have looked at three different styles for proving such statements. Study the lecture notes carefully.

OBS.

- A statement like (1) tells us *only* that there is a *relationship* between \mathcal{P} and \mathcal{Q} , but it does **not** tell us that those two statements are actually true on their own.
- Mathematical proofs are not concerned with the *absolute truth* of a given statement. They are merely used to deduce a piece of information from another. This is true even when the statement we want to prove is not explicitly in the form (1), but any mathematical statement involves a hypothesis.
- Note that (1) is not the same as

$$\boxed{\text{If } \mathcal{Q} \text{ then } \mathcal{P}}. \quad (2)$$

This statement is known as the **converse** of (1). It is obtained by exchanging the places of \mathcal{P} and \mathcal{Q} in the original statement. The statements (1) and (2) are **not** equivalent. The question of whether a converse is true or false is a *separate issue* from whether the original statement is true or false.

- The statement

$$\boxed{\mathcal{P} \text{ if and only if } \mathcal{Q}}. \quad (3)$$

is used to mean

$$\boxed{\text{If } \mathcal{P} \text{ then } \mathcal{Q}} \quad \text{and} \quad \boxed{\text{If } \mathcal{Q} \text{ then } \mathcal{P}}.$$

This boils down to the assertion that \mathcal{P} and \mathcal{Q} are **equivalent**, i.e. they are, in each case that can arise, either *both* true or *both* false.

Notation: $\mathcal{P} \iff \mathcal{Q}$ is read “ \mathcal{P} is equivalent to \mathcal{Q} ” and means exactly the same as the statement (3) above.

OBS.

- To prove a statement like (3), it is sometimes possible to do it in one go by using a *Style #2* proof with “ \implies ” replaced by “ \iff ”. More usually, it is necessary to divide the proof into two parts, and prove $\mathcal{P} \implies \mathcal{Q}$ and $\mathcal{Q} \implies \mathcal{P}$ separately.
- A proof of a statement like (3) only establishes a *relationship* between \mathcal{P} and \mathcal{Q} . It does **not** prove that these two statements are actually both true. Further information would be required to determine this.

- **Examples and counterexamples**

Consider the following statement:

$$\boxed{\text{If } a, b, c \in \mathbb{R} \text{ then } a^2 + b^2 + c^2 \geq ab + bc + ac.}$$

This is a true statement, as can be seen from the proof included below.

Proof: Since a, b, c are real numbers, it follows that

$$(a - b)^2 \geq 0, \quad (b - c)^2 \geq 0, \quad (c - a)^2 \geq 0.$$

Expanding the left-hand sides of these inequalities and adding them up, we get

$$2(a^2 + b^2 + c^2 - ab - ac - bc) \geq 0,$$

and the conclusion of the above statement follows immediately. There are several points to note about this proof.

1. This result was proved by considering three *arbitrary* real numbers, a, b , and c . That is typical: to prove that a result is true, it is necessary to ensure that **all** cases are covered by considering the *most general case*.
2. The observation that, say, $a = 1, b = 1$ and $c = 0$ satisfy

$$1^2 + 1^2 + 0^2 \geq 1 \times 1 + 1 \times 0 + 1 \times 0$$

may be a useful **example** or **illustration** of the result we proved above, but it does not constitute a proof of it. A proof must consider all possibilities for $a, b, c \in \mathbb{R}$.

3. However, if we are trying to prove that a given statement is **false**, a simple example may be enough to do it. For example, to show that the statement

$$\boxed{\text{If } a, b, c \in \mathbb{R} \text{ then } a^2 + b^2 + c^2 < ab + bc + ac}$$

is **false**, it is sufficient to observe that, if $a = 1, b = 0, c = 0$, then

$$\text{LHS} = 1^2 + 0^2 + 0^2 = 1 \quad \text{and} \quad \text{RHS} = 1 \times 0 + 1 \times 0 + 0 \times 0 = 0,$$

which clearly shows that our statement is false.

4. An example like the one above that is produced to demonstrate that a statement is false is called a **counterexample** to the statement.

• **Proof by contradiction** This works as follows.

1. Suppose that we wish to prove the truth of some mathematical proposition Q .
2. We start by **assuming the contrary**, i.e. assuming Q false.
3. From that assumption (which represents an extra hypothesis), we argue towards a **contradiction**, i.e. a conclusion that is *clearly* absurd, being either contrary to known facts or self-contradictory.
4. An absurd conclusion obtained by correct reasoning reveals that we must have introduced an erroneous assumption. But the only questionable assumption we have introduced is that Q is false. *That assumption is therefore wrong.*
5. So, on obtaining the “contradiction”, we are entitled to conclude that the original proposition Q is true.

The next example should be studied carefully. All proofs by contradiction follow the same pattern.

Example: Let A, B be $n \times n$ matrices. Prove that if $AB = O$ but $B \neq O$, then A is singular (i.e. does not have an inverse).

Solution

STEP 1 Suppose that

$$AB = O \quad \text{but} \quad B \neq O. \tag{4}$$

We must show that A is singular.

STEP 2 Assume the contrary, i.e. assume that A is non-singular.

STEP 3 Then A^{-1} exists and, on pre-multiplying the first relation in (4) by A^{-1} , we find successively

$$\begin{aligned} A^{-1}(AB) &= A^{-1}O, \\ \text{i.e. } IB &= O, \\ \text{i.e. } B &= O, \end{aligned}$$

a contradiction, since $B \neq O$ according to the second relation in (4).

STEP 4 It follows that A must be non-singular and this completes the proof.

OBS. Note very carefully the style: the *hypotheses* of the statement to be proved come first (STEP 1), **unchanged**, before we assume the negation of the conclusion (STEP 2). Also, the proof should end with a sentence that states that the original assumptions was wrong and therefore the conclusion is true.