Mathematics 1T (Algebra)

Summary of Week #5

Notation: In this section t will denote *time* measured from some initial moment, which we agree to take $t = 0$. Y denotes a function of time, i.e. $Y = Y(t)$, but for the sake of brevity the dependence on t will not be recorded explicitly.

- **Definition of exponential growth/decay** Let A and k be two positive constants. If a quantity Y changes over time in accordance with the law

$$Y = A\,e^{kt} \qquad \text{or} \qquad Y = A\,e^{-kt}$$

  then Y is said to have *exponential growth* or *exponential decay*, respectively.

- **How to identify exponential growth/decay** There are two ways to achieve this:

  1. The rate of change of Y at time t is proportional to the amount of Y present at that time. More precisely,

     (a) A quantity Y has *exponential growth* if and only if

     $$\frac{dY}{dt} = kY,$$

     for some $k > 0$.

     (b) A quantity Y has *exponential decay* if and only if

     $$\frac{dY}{dt} = -kY,$$

     for some $k > 0$.

  2. A quantity Y has exponential growth (decay) if and only if the graph of $\ln Y$ against t is a straight line with positive (negative) gradient.

- **Properties + terminology**:

  1. When $Y = A\,e^{\pm kt}$, then

     $$A = \text{the initial value of } Y.$$

  2. Suppose that Y has exponential growth/decay. Then the value of Y at the end of an interval of time, say, $I = (t_0, t_0 + \ell)$, when expressed as a fraction, multiple or percentage of the value of Y at the start of the interval depends only on the length of the interval (i.e. ℓ) and not on the time at which the interval started (i.e. t_0).

  3. If Y has exponential decay, the length of time it takes for Y to reduce to one-half of its initial size is known as its **half-life**.

4. If Y has exponential decay described by a law of the form
$$Y = A\,\mathrm{e}^{-kt}, \qquad (A,\ k \text{ positive})$$
then
$$\boxed{\text{half-life of } Y = \frac{\ln 2}{k}}\ .$$

5. If Y has exponential growth described by a law of the form
$$Y = A\,\mathrm{e}^{kt}, \qquad (A,\ k \text{ positive})$$
then
$$\boxed{\text{the time } Y \text{ takes to double} = \frac{\ln 2}{k}}\ .$$

6. If Y is growing exponentially, and increases by $r\%$ in one unit of time, where r is small compared with 100, then
$$\boxed{\text{the time } Y \text{ takes to double} \approx \frac{70}{r}}\ .$$

Aside

- Laws of exponentials:
$$\mathrm{e}^A \times \mathrm{e}^B = \mathrm{e}^{A+B}$$
$$\left(\mathrm{e}^A\right)^B = \mathrm{e}^{AB}$$
$$\frac{\mathrm{e}^A}{\mathrm{e}^B} = \mathrm{e}^{A-B}$$
$$\mathrm{e}^{-A} = \frac{1}{\mathrm{e}^A}$$
$$\mathrm{e}^0 = 1$$
$$\mathrm{e}^1 \approx 2.718$$

for all $A,\ B \in \mathbb{R}$.

- Natural logarithms are defined by the relationship
$$\boxed{\text{If} \quad x = \mathrm{e}^y \quad (x > 0) \qquad \text{then} \qquad y = \ln x}\ .$$

Laws of logarithms (valid for logarithms to any base):
$$\ln(A \times B) = \ln A + \ln B$$
$$\ln\left(\frac{A}{B}\right) = \ln A - \ln B$$
$$\ln\left(\frac{1}{B}\right) = -\ln B$$
$$\ln\left(A^\alpha\right) = \alpha \ln A$$
$$\ln 1 = 0$$
$$\ln\left(\mathrm{e}^A\right) = A$$
$$\mathrm{e}^{\ln A} = A$$

for all $A,\ B > 0$ and $\alpha \in \mathbb{R}$.