Mathematics 1T (Algebra)

Summary of Week #4

- **Intersection of two planes** In 3D, any two planes are either parallel (in which case they have parallel *normal vectors*), or they intersect in a line.

The following example should be studied carefully.
Example: Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the planes

$$\mathcal{P}_1: \quad x - 2y + 3z = 4,$$

$$\mathcal{P}_2: \quad 2x + y + z = 3.$$

Find the symmetric form equations for the line of intersection of these two planes.

Solution The required line of intersection is the locus of points $P(x,\,y,\,z)$ satisfying simultaneously the equations

$$\begin{cases} x - 2y + 3z = 4, \\ 2x + y + z = 3. \end{cases} \tag{1}$$

This represents a system of two equations in three unknowns (x, y, and z) which must be solved using the knowledge gained in *Mathematics 1R*. The augmented matrix of this system is

$$\begin{bmatrix} 1 & -2 & 3 & 4 \\ 2 & 1 & 1 & 3 \end{bmatrix}$$

and, by performing elementary row operations on this matrix, we obtain its reduced echelon form

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \end{bmatrix}.$$

Therefore, the linear system of equations (1) is equivalent to

$$\begin{cases} x + z = 2, \\ y - z = -1, \end{cases}$$

and hence has general solution

$$z = t, \qquad y = -1 + z = -1 + t, \qquad x = 2 - z = 2 - t.$$

In conclusion, we have found that the line of intersection of the two planes, $\mathcal{P}_1$ and $\mathcal{P}_2$, has parametric equations

$$\begin{cases} x = 2 - t, \\ y = -1 + t, \qquad (t \in \mathbb{R}), \\ z = t. \end{cases}$$

The symmetric form of the same line follows immediately by eliminating t (the parameter),

$$\frac{x-2}{-1} = \frac{y+1}{1} = \frac{z}{1}.$$

- **Angles** Suppose that $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors. The *angle* between $\mathbf{a}$ *and* $\mathbf{b}$ is the angle in $[0,\pi]$ between any two representatives with a common starting point (see below).

  OBS.
  1. $\theta = 0$ means that $\mathbf{a}$ and $\mathbf{b}$ are in the same direction.

  2. $\theta = \pi$ means that $\mathbf{a}$ and $\mathbf{b}$ are in the opposite direction.

  3. $\theta = \pi/2$ means that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

  4. the angle between $\mathbf{a}$ and $\mathbf{b}$ is the same as the angle between $\mathbf{b}$ and $\mathbf{a}$, *i.e* no sense is attached to the angle.
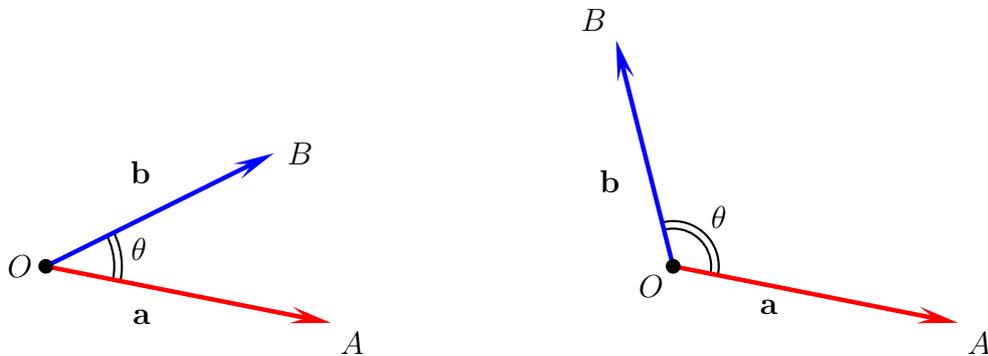
- **Products** This week we introduced two ways of "multiplying" vectors:

  - **Scalar product:** For any two vectors $\mathbf{a}$ and $\mathbf{b}$, their scalar product is denoted by

    $$\mathbf{a}\cdot\mathbf{b}$$

  1. It is defined to be a *scalar* calculated according to

  $$\mathbf{a}\cdot\mathbf{b} = \begin{cases} |\mathbf{a}||\mathbf{b}|\cos\theta, & \text{if} \quad \mathbf{a} \text{ and } \mathbf{b} \text{ are non-zero,} \\ 0 & \text{if} \quad \mathbf{a} \text{ or } \mathbf{b} \text{ (or both) are zero,} \end{cases} \tag{2}$$

  where θ is the angle between the two vectors.

  2. If $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, then

  $$\mathbf{a}\cdot\mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{3}$$

3. Using (2) and (3), we can find the angle θ between two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$

$$\boxed{\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}} \tag{4}$$

Taking into account that

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} \qquad \text{and} \quad |\mathbf{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$$

(see *Summary of Week #2*), this formula can be written as

$$\cos\theta = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

4. Algebraic properties of scalar products (see *Handout #3*).

− **Vector product:** For any two vectors $\mathbf{a}$ and $\mathbf{b}$, their vector product is denoted by

$$\boxed{\mathbf{a}\times\mathbf{b}}$$

1. If $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, it is defined to be a *vector* whose component are

$$\boxed{\mathbf{a}\times\mathbf{b} = (a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1)}$$

2. Every vector product $\mathbf{a}\times\mathbf{b}$, if non-zero, is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

• **Applications**

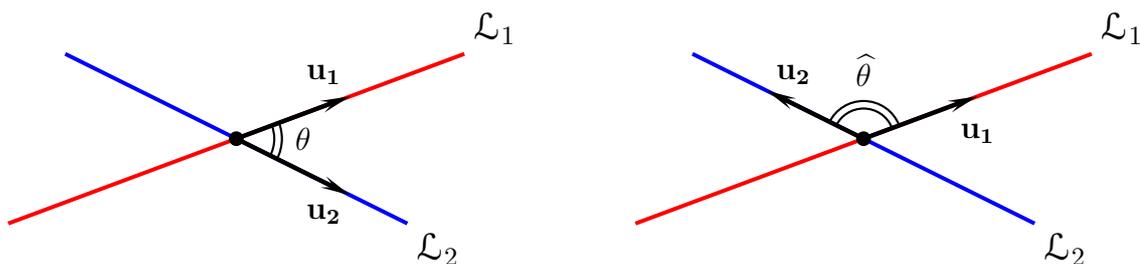1. **Angles between two intersecting lines** ≡ *angles between direction vectors for the lines*.

(a) Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two intersecting lines with <u>direction vectors</u> $\mathbf{u_1}$ and $\mathbf{u_2}$, respectively. As illustrated below, there are two possible choices for the angle between the vectors $\mathbf{u_1}$ and $\mathbf{u_2}$. Using formula (4) will give either the *acute* angle, θ, or the *obtuse* angle, $\widehat{\theta}$. However, note that

$$\theta + \widehat{\theta} = \pi,$$

so that if one of the two angles is specifically required and the other happens to appear, then there is no real problem. The correct value is simply obtained by subtracting from π the angle found (see the examples discussed in class).

(b) The angle between $\mathcal{L}_1$ and $\mathcal{L}_2$ (that is, either θ or $\widehat{\theta}$) is given by
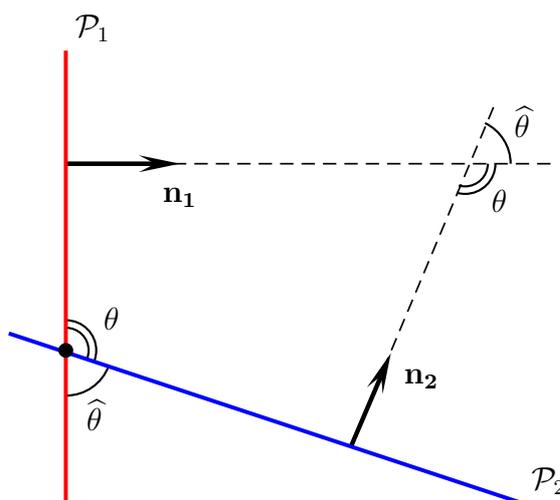
$$\boxed{\cos^{-1}\left(\frac{\mathbf{u_1}\cdot\mathbf{u_2}}{|\mathbf{u_1}||\mathbf{u_2}|}\right)}$$

2. **Angles between two intersecting planes** ≡ *angles between normal vectors of the planes*.

   (a) The same remarks as above apply to this case as well.
   (b) If the two planes $\mathcal{P}_1$ and $\mathcal{P}_2$ have <u>normal vectors</u> $\mathbf{n}_1$ and $\mathbf{n}_2$, respectively, then the angle between the two planes (that is, either θ or $\widehat{\theta}$) is given by

$$\cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right)$$

3. **Finding the normal vector to a plane through three given points** The question here is to determine the equation of the plane, say $\mathcal{P}$, through three given points

$$P(x_P,\, y_P,\, z_P), \qquad Q(x_Q,\, y_Q,\, z_Q), \qquad R(x_R,\, y_R,\, z_R).$$

Recall from *Week #3* that the equation of a plane is completely determined once we know a particular point in that plane *and* a vector perpendicular to it. In this particular case, the candidates for these two quantities can be chosen as follows:

   – *particular point:* any of the three given points will serve our purpose equally well.

– *normal vector:* this requires a bit of thinking. In standard notation,

$$\overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = (x_Q - x_P,\ y_Q - y_P,\ z_Q - z_P),$$
$$\overrightarrow{PR} = \mathbf{r} - \mathbf{p} = (x_R - x_P,\ y_R - y_P,\ z_R - z_P).$$

The vector product $\overrightarrow{PQ} \times \overrightarrow{PR}$ will be a vector perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, and so will be a normal vector for $\mathcal{P}$, etc.

Numerical example: Determine the plane $\mathcal{P}$ through the points $P(1, -2, 3)$, $Q(1, 1, 1)$, and $R(0, 1, -1)$.

Solution A normal vector to $\mathcal{P}$ is

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR}. \tag{5}$$

The two terms in this vector product can be calculated explicitly,

$$\overrightarrow{PQ} = (1,\ 1,\ 1) - (1,\ -2,\ 3) \quad = (0,\ 3,\ -2), \tag{6a}$$
$$\overrightarrow{PR} = (0,\ 1,\ -1) - (1,\ -2,\ 3) = (-1,\ 3,\ -4). \tag{6b}$$

From (5) and (6) we find

$$\mathbf{n} = (0,\ 3,\ -2) \times (-1,\ 3,\ -4) = (-6,\ 2,\ 3).$$

Next, use equation (2) in *Summary of Week #3* with $(A,\ B,\ C) \equiv (-6,\ 2,\ 3)$ and $(x_0,\ y_0,\ z_0) \equiv (1,\ -2,\ 3)$. Therefore, we find that $\mathcal{P}$ is described by the equation

$$(-6)(x - 1) + 2(y - (-2)) + 3(z - 3) = 0, \tag{7}$$

or

$$\boxed{-6x + 2y + 3z = -1}\,.$$

OBS. We could have used

$$(x_0,\ y_0,\ z_0) \equiv \underbrace{(1,\ 1,\ 1)}_{Q} \qquad \text{or} \qquad (x_0,\ y_0,\ z_0) \equiv \underbrace{(0,\ 1,\ -1)}_{R}.$$

Also, a normal vector can be obtained by considering

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{QR} \qquad \text{or} \qquad \mathbf{n} = \overrightarrow{PR} \times \overrightarrow{QR},$$

etc. *The final result must be the same whatever the choice made.*