

# Mathematics 1T (Algebra)

## Summary of Week #3

- **General planes in three dimensional space**

1. *Planes* are generally described by specifying
  - (a) a **normal vector** (i.e. a vector perpendicular to the plane).
  - (b) a **point** that is in the plane.
2. The general equation

$$Ax + By + Cz = D \tag{1}$$

defines a plane that has  $\mathbf{n} = (A, B, C)$  as a normal vector.

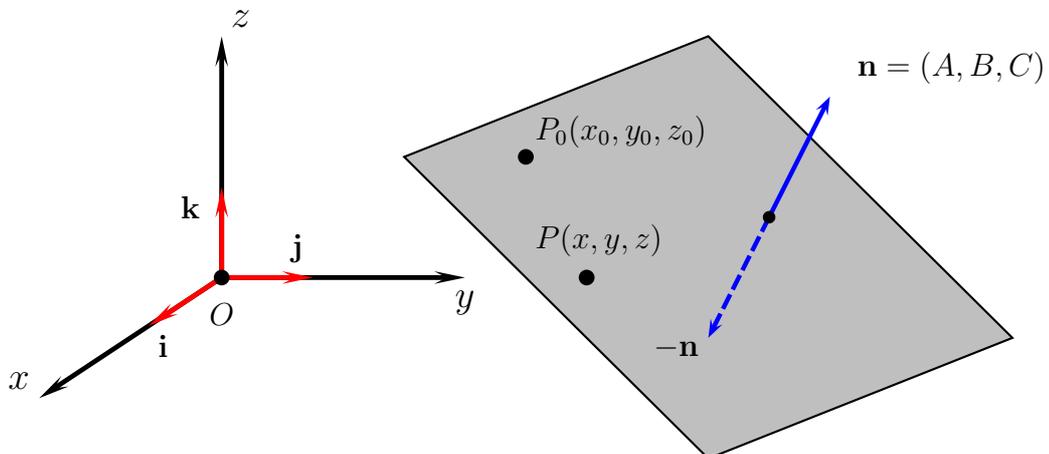
**OBS.**

- The constant  $D$  that appears in (1) can be found immediately once we know the coordinates of a particular point in that plane, say,  $P_0(x_0, y_0, z_0)$ .
- If  $\mathbf{u}$  is a normal vector for a given plane, then so is  $\lambda\mathbf{u}$  for any  $\lambda \in \mathbb{R}, \lambda \neq 0$ .

3. The equation

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \tag{2}$$

defines a plane through  $P_0(x_0, y_0, z_0)$ , and which has  $\mathbf{n} = (A, B, C)$  as a normal vector.



• **Lines in three dimensional space**

1. *Lines* can be described by specifying
  - (a) a **direction vector** (i.e. a non-zero vector parallel to the line).
  - (b) a **point** that is on the line.
2. Let  $A(x_A, y_A, z_A)$  be an arbitrary point in space, and  $\mathbf{u} = (l, m, n)$  a non-zero vector. The equation of the line  $\mathcal{L}$  through  $A$  with direction vector  $\mathbf{u}$  can be expressed in three different ways:
  - (a) **Vector form:** The position vector  $\mathbf{r}$  of an arbitrary point on  $\mathcal{L}$  is described by the equation

$$\mathbf{r} = \mathbf{a} + t\mathbf{u} \quad (t \in \mathbb{R}), \quad (3)$$

where  $\mathbf{a}$  is the position vector of  $A$  and  $t \in \mathbb{R}$  is called the *parameter*.

- (b) **Parametric form:** This is obtained by simply re-writing (3) on components. If the coordinates of a generic point on  $\mathcal{L}$  are  $(x, y, z)$ , then

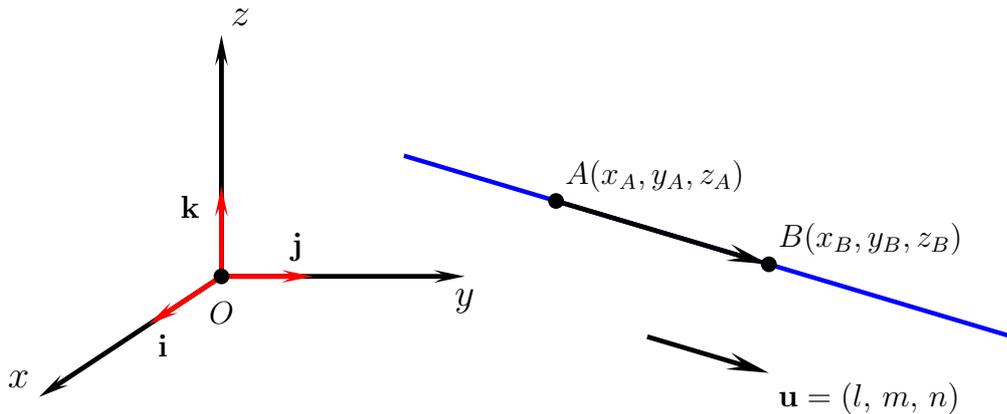
$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

and, since  $\mathbf{a} = x_A\mathbf{i} + y_A\mathbf{j} + z_A\mathbf{k}$ , equation (3) leads to

$$x = x_A + tl, \quad y = y_A + tm, \quad z = z_A + tn \quad (t \in \mathbb{R}). \quad (4)$$

- (c) **Symmetric form:** If  $l, m$ , and  $n$  are all non-zero, we may eliminate the parameter  $t$  to obtain

$$\frac{x - x_A}{l} = \frac{y - y_A}{m} = \frac{z - z_A}{n}. \quad (5)$$



**OBS.**

- In (5), the denominators give the components of a direction vector for  $\mathcal{L}$ .
- To get back from (5) to (4), note that the three equal expressions in the symmetric form are in fact *all equal to the parameter*.
- If  $A(x_A, y_A, z_A)$  and  $B(x_B, y_B, z_B)$  are two distinct points on  $\mathcal{L}$ , then  $\overrightarrow{AB}$  is a direction vector for  $\mathcal{L}$ . In *Week #2* we have seen that

$$\overrightarrow{AB} = (x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k},$$

or, in other words,  $\overrightarrow{AB} = (x_B - x_A, y_B - y_A, z_B - z_A)$ . Hence, the equations in symmetric form for a line  $\mathcal{L}$  through two given points  $A$  and  $B$ , as above, can be stated in the form

$$\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A} = \frac{z - z_A}{z_B - z_A}. \quad (6)$$

- *Given either the parametric or symmetric form for a line  $\mathcal{L}$ , we may determine a direction vector  $(l, m, n)$  and a point  $(x_A, y_A, z_A)$  on  $\mathcal{L}$  by inspection.*

- **Intersections** A pair of distinct lines in  $2D$  are either parallel and never intersect, or are non-parallel and have a unique point of intersection.

In  $3D$  things are rather different, since *a pair of distinct lines in space may be non-parallel but still have no point of intersection*.

Finding whether or not two given lines intersect is based on the use of the parametric equations for the two lines. The following example should be studied carefully.

**Example:** Let  $\mathcal{L}_1$  and  $\mathcal{L}_2$  be the lines

$$\mathcal{L}_1 : \frac{x + 3}{2} = \frac{y}{-1} = \frac{z + 2}{3},$$
$$\mathcal{L}_2 : \frac{x - 2}{1} = \frac{y - 2}{4} = \frac{z - 5}{1}.$$

Prove that these two lines intersect and find their point of intersection.

*Solution*

**STEP 1** Find the parametric form for the equations of  $\mathcal{L}_1$  and  $\mathcal{L}_2$ . To avoid confusion, we must use **different letters** for the parameters of **different lines**

$t$  = parameter for  $\mathcal{L}_1$

$s$  = parameter for  $\mathcal{L}_2$

In parametric form, the two lines are described by the following equations

$$\mathcal{L}_1 : \begin{cases} x = 2t - 3 \\ y = -t \\ z = 3t - 2 \end{cases} \quad \mathcal{L}_2 : \begin{cases} x = s + 2 \\ y = 4s + 2 \\ z = s + 5 \end{cases}$$

**STEP 2** The lines  $\mathcal{L}_1$  and  $\mathcal{L}_2$  intersect if and only if there is a point of  $\mathcal{L}_1$  that is also a point of  $\mathcal{L}_2$ . The intersection point, if any, must lie simultaneously on both lines. Using the parametric form equations stated above, this happens if and only if there exist  $t, s \in \mathbb{R}$  such that

$$\begin{cases} 2t - 3 = s + 2 \\ -t = 4s + 2 \\ 3t - 2 = s + 5, \end{cases}$$

which reduces to

$$\begin{cases} 2t - s = 5 \\ -t - 4s = 2 \\ 3t - s = 7. \end{cases} \quad (7)$$

**STEP 3** Note that (7) represents a system of three equations in two unknowns,  $t$  and  $s$ . If this system is **consistent** then the two lines have a common point, otherwise they are non-intersecting lines.

In this particular case the system turns out to be consistent, and the solution is

$$t = 2 \quad \text{and} \quad s = -1.$$

**STEP 4** Finally, the last step consists in getting the coordinates of the intersection point. This can be achieved in two ways:

1. the intersection point is the point on  $\mathcal{L}_1$  with  $t = 2$ :

$$x = 2 \times 2 - 3 = 1, \quad y = -2, \quad z = 3 \times 2 - 2 = 4,$$

i.e. the intersection point is  $(1, 2, -4)$ .

2. the intersection point is also the point on  $\mathcal{L}_2$  with  $s = -1$ :

$$x = (-1) + 2 = 1, \quad y = 4 \times (-1) + 2 = -2, \quad z = (-1) + 5 = 4, \text{ etc.}$$