Mathematics 1T (Algebra)

Summary of Week #11

- **Total Probability**

    1. This concerns the situation where we have an event $\mathcal{X}$ that we arrive at via any one of the n intermediate mutually exclusive events $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$.

    2. Examples of such situations:

        (a) A pair of shoes may be defective, having been manufactured at one of a company's three factories, A, B or C.

        (b) A person may have lung cancer, having been either a smoker or a non-smoker.

        (c) A student may pass Mathematics–1T, having obtained grade A, B or C in SCE Higher Mathematics.

    3. **Law of Total Probability:** Let $\mathcal{X}$ be an event that arises as a result of any one of the mutually exclusive events $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$. Then,

$$\boxed{P(\mathcal{X}) = P(\mathcal{A}_1)P(\mathcal{X} \,|\, \mathcal{A}_1) + P(\mathcal{A}_2)P(\mathcal{X} \,|\, \mathcal{A}_2) + \cdots + P(\mathcal{A}_n)P(\mathcal{X} \,|\, \mathcal{A}_n)} \,. \qquad (1)$$

Example #1:

A company has three factories, A, B, C which produce, respectively, 50%, 30%, 20% of its output of compact discs. It has been found that the percentage of defective discs among the outputs of factories A, B, C are, respectively, 4%, 2%, 1%.

A disc is chosen at random from the company's total output. What is the probability that is defective?

Solution

Define the following events,

$$\begin{aligned}
\mathcal{X} &= \text{the disc chosen is defective,} \\
\mathcal{A}_1 &= \text{the disc was manufactured by } A, \\
\mathcal{A}_2 &= \text{the disc was manufactured by } B, \\
\mathcal{A}_3 &= \text{the disc was manufactured by } C.
\end{aligned}$$

From the first sentence in the statement of the problem above, we have that

$$P(\mathcal{A}_1) = \frac{50}{100}, \qquad P(\mathcal{A}_2) = \frac{30}{100}, \qquad P(\mathcal{A}_3) = \frac{20}{100}. \qquad (2)$$

Next, we notice that $P(\mathcal{X} \,|\, \mathcal{A}_1)$ represents the probability that a disc manufactured by factory A is defective; similar statements hold for $P(\mathcal{X} \,|\, \mathcal{A}_2)$ and $P(\mathcal{X} \,|\, \mathcal{A}_3)$. Thus, according to the information provided in the hypothesis,

$$P(\mathcal{X} \,|\, \mathcal{A}_1) = \frac{4}{100}, \qquad P(\mathcal{X} \,|\, \mathcal{A}_2) = \frac{2}{100}, \qquad P(\mathcal{X} \,|\, \mathcal{A}_3) = \frac{1}{100}. \qquad (3)$$

In conclusion, by applying (1), we obtain the required answer,

$$P(\mathcal{X}) = P(\mathcal{A}_1)P(\mathcal{X}\,|\,\mathcal{A}_1) + P(\mathcal{A}_2)P(\mathcal{X}\,|\,\mathcal{A}_2) + P(\mathcal{A}_3)P(\mathcal{X}\,|\,\mathcal{A}_3)$$
$$= \frac{50}{100} \times \frac{4}{100} + \frac{30}{100} \times \frac{2}{100} + \frac{20}{100} \times \frac{1}{100}$$
$$= \frac{28}{1000} = 0.028.$$

Conditional probabilities: "the wrong way round"   Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two arbitrary events. In some problems it is useful to know the following formula relating the conditional probabilities of these events,

$$P(\mathcal{A}_1\,|\,\mathcal{A}_2) = \frac{P(\mathcal{A}_1)P(\mathcal{A}_2\,|\,\mathcal{A}_1)}{P(\mathcal{A}_2)}. \qquad (4)$$

Example #2:

Among 50–year-olds the ratio of smokers to non-smokers is 1 to 2. It is found that among smokers of this age, 40% suffer from a disease D, whereas only 10% of non-smoking 50–year-olds suffer from D. An individual is selected at random from the population of 50–year-olds. What are:

(i). the probability that he suffers from D?

(ii). the probability that he is a smoker and suffers from D?

(iii). the probability that he suffers from D given that he is a smoker?

(iv). the probability that he is a smoker given that he suffers from D?

Solution

We start with a simple observation. Since we are given that the ratio smokers/non-smokers is equal to 1/2, it follows that

- 1/3 of the population consists of *smokers*,

- 2/3 of the population is made up of *non-smokers*.

As usually, define the corresponding events appearing in the example above,

$$\mathcal{X} = \text{an individual suffers from } D,$$
$$\mathcal{A}_1 = \text{individual is a } smoker ,$$
$$\mathcal{A}_2 = \text{individual is a } non\text{-}smoker .$$

Clearly, the answer to (i) involves finding $P(\mathcal{X})$, which requires a straightforward application of the *Law of Total Probability* (1):

$$P(\mathcal{X}) = P(\mathcal{A}_1)P(\mathcal{X}\,|\,\mathcal{A}_1) + P(\mathcal{A}_2)P(\mathcal{X}\,|\,\mathcal{A}_2) \qquad (5)$$

Note that we are provided with the following information in the hypothesis

$$P(\mathcal{A}_1) = \frac{1}{3}, \qquad P(\mathcal{A}_2) = \frac{2}{3} , \qquad (6\text{a})$$
$$P(\mathcal{X}\,|\,\mathcal{A}_1) = \frac{40}{100}, \qquad P(\mathcal{X}\,|\,\mathcal{A}_2) = \frac{10}{100}, \qquad (6\text{b})$$

Substituting these values in (5), we are led to

$$\text{answer} = \frac{1}{3} \times \frac{40}{100} + \frac{2}{3} \times \frac{10}{100} = \frac{1}{5} \,.$$

(ii). We are asked to calculate

$$P(\mathcal{A}_1 \cap \mathcal{X}) \,,$$

which can be found with the help of equation (5) from *Summary of Week* #10, together with (6b) above,

$$P(\mathcal{A}_1 \cap \mathcal{X}) = P(\mathcal{A}_1)P(\mathcal{X} \,|\, \mathcal{A}_1)$$
$$= \frac{1}{3} \times \frac{40}{100} = \frac{2}{15} \,.$$

(iii). The answer is

$$P(\mathcal{X} \,|\, \mathcal{A}_1) = \frac{40}{100} = \frac{2}{5} \,.$$

(iv). We are asked to calculate

$$P(\mathcal{A}_1 \,|\, \mathcal{X}) \,.$$

The easiest way to find this number is by using (4),

$$P(\mathcal{A}_1 \,|\, \mathcal{X}) = \frac{P(\mathcal{A}_1)P(\mathcal{X} \,|\, \mathcal{A}_1)}{P(\mathcal{X})} \,, \tag{7}$$

and we notice that all the probabilities involved in the RHS of (7) are known. Hence,

$$P(\mathcal{A}_1 \,|\, \mathcal{X}) = \frac{\frac{1}{3} \times \frac{2}{5}}{\frac{1}{5}} = \frac{2}{3} \,.$$

- **The Binomial Pattern**

   1. This is the name for the pattern that occurs when we repeat an experiment n times and look at how often a particular event occurred.

   2. Examples of such situations:

   (a) **Example #1**   A dice is thrown 6 times. What is the probability that 6 comes up exactly twice?

   (b) **Example #2**   A factory produces ball-point pens and packages them in boxes of 5 pens. For each pen coming off the production line, the probability of it being defective is 10%. What is the probability that a box of 5 pens contains exactly 3 defective pens?

   (c) **Example #3**   20% of the cars travelling on a certain motorway have defective tyres. One day the police stop at random 8 cars using the motorway and inspect their tyres.
   What is the probability that they find at least 2 cars with defective tyres?

3. **Main Theorem:** Let $\mathcal{A}$ be an event associated with an experiment $\mathcal{E}$. Suppose that this experiment is performed n times and that the n performances are independent of each other (i.e. the result of one performance does not affect the result of any other). Suppose also that each time $\mathcal{E}$ is performed, $P(\mathcal{A}) = p$. Then,

$$P \left(\begin{matrix} \mathcal{A} \text{ occurs exactly } k \text{ times in} \\ n \text{ performances of } \mathcal{E} \end{matrix} \right) = \binom{n}{k} p^k (1-p)^{n-k}. \tag{8}$$

$\boxed{\text{OBS.}}$ Note that (8) can be applied very widely. It can be used whenever a routine is repeated with *unchanged probabilities*:

- repeatedly tossing a coin,
- testing items from a production line for defects,
- drawing articles from a bag (*with replacement*).

Solution of **Example #1**:

To apply (8), we need to identify what are $\mathcal{E}$ and $\mathcal{A}$, as well as the numbers n, k and p involved in that formula. For this particular example, notice that

$$\mathcal{E} = \text{a dice is thrown once,}$$
$$\mathcal{A} = \text{the number obtained is } 6.$$

Also, on each throw, $P(\mathcal{A}) = 1/6$, the dice is thrown 6 times, and we are interested in the occurrence of $\mathcal{A}$ twice. Hence,

$$p = \frac{1}{6}, \qquad n = 6, \qquad \text{and} \quad k = 2. \tag{9}$$

On using (9) in (8), we find

$$P \left(\begin{matrix} \mathcal{A} \text{ occurs exactly 2 times} \\ \text{in 6 performances of } \mathcal{E} \end{matrix} \right) = \binom{6}{2} \times \left(\frac{1}{6} \right)^2 \times \left(\frac{5}{6} \right)^4 \approx 0.201.$$

Solution of **Example #2**:

In this case,

$$\mathcal{E} = 1 \text{ pen is chosen at random from the production line,}$$
$$\mathcal{A} = \text{the pen chosen is defective}.$$

The experiment is repeated 5 times (since each box contains 5 pens), and we are given in the hypothesis that $P(\mathcal{A}) = 0.1$. Hence,

$$p = 0.1, \qquad n = 5, \qquad \text{and} \quad k = 3. \tag{10}$$

On using (10) in (8), we find

$$P \left(\begin{matrix} \mathcal{A} \text{ occurs exactly 3 times} \\ \text{in 5 performances of } \mathcal{E} \end{matrix} \right) = \binom{5}{3} \times (0.1)^3 \times (0.9)^2 \approx 0.114.$$

Solution of **Example #3**:

This problem is slightly more difficult. We start off with identifying $\mathcal{E}$ and $\mathcal{A}$:

$$\mathcal{E} = \text{the police stop at random a car and inspect its tyres,}$$
$$\mathcal{A} = \text{car has defective tyres}.$$

Next thing, we must recognise that the situation involved here,

$$\mathcal{B} = \text{``at least 2 cars with defective tyres''},$$

can be broken down into a number of simpler situations. To this end, define the following events

$$\mathcal{B}_j = j \text{ cars with defective tyres} \qquad (j = 2, 3, \ldots, 8),$$

and notice that $\mathcal{B}$ is actually

$$\mathcal{B}_2 \cup \mathcal{B}_3 \cup \cdots \cup \mathcal{B}_8. \tag{11}$$

In order to calculate the probability of this event, we use the fact that the events involved in (11) are mutually exclusive, and thus

$$P(\mathcal{B}_2 \cup \mathcal{B}_3 \cup \cdots \cup \mathcal{B}_8) = P(\mathcal{B}_2) + P(\mathcal{B}_3) + \cdots + (\mathcal{B}_8). \tag{12}$$

All that remains to be done is evaluate the individual probabilities on the RHS of (12); this will require the application of (8).

It is given in the hypothesis that $P(\mathcal{A}) = 20/100 = 1/5$. Also, since 8 cars are inspected, this means that $\mathcal{E}$ (defined above) is repeated 8 times. Without going into any further details,

$$P(\mathcal{B}_2) = \binom{8}{2} \times \left(\frac{1}{5}\right)^2 \times \left(\frac{4}{5}\right)^6 \qquad n = 8, \ k = 2, \ p = \frac{1}{5},$$

$$P(\mathcal{B}_3) = \binom{8}{3} \times \left(\frac{1}{5}\right)^3 \times \left(\frac{4}{5}\right)^5 \qquad n = 8, \ k = 3, \ p = \frac{1}{5},$$

$$P(\mathcal{B}_4) = \binom{8}{4} \times \left(\frac{1}{5}\right)^4 \times \left(\frac{4}{5}\right)^4 \qquad n = 8, \ k = 4, \ p = \frac{1}{5},$$

$$P(\mathcal{B}_5) = \binom{8}{5} \times \left(\frac{1}{5}\right)^5 \times \left(\frac{4}{5}\right)^3 \qquad n = 8, \ k = 5, \ p = \frac{1}{5},$$

$$P(\mathcal{B}_6) = \binom{8}{6} \times \left(\frac{1}{5}\right)^6 \times \left(\frac{4}{5}\right)^2 \qquad n = 8, \ k = 6, \ p = \frac{1}{5},$$

$$P(\mathcal{B}_7) = \binom{8}{7} \times \left(\frac{1}{5}\right)^7 \times \left(\frac{4}{5}\right)^1 \qquad n = 8, \ k = 7, \ p = \frac{1}{5},$$

$$P(\mathcal{B}_8) = \binom{8}{8} \times \left(\frac{1}{5}\right)^8 \times \left(\frac{4}{5}\right)^0 \qquad n = 8, \ k = 8, \ p = \frac{1}{5}.$$

The final answer is obtained by substituting these probabilities in (12), leading us to

$$P(\mathcal{B}_2 \cup \mathcal{B}_3 \cup \cdots \cup \mathcal{B}_8) = \sum_{r=2}^{8} \binom{8}{r} \times \left(\frac{1}{5}\right)^r \times \left(\frac{4}{5}\right)^{8-r}$$

$$= \sum_{r=0}^{8} \binom{8}{r} \times \left(\frac{1}{5}\right)^r \times \left(\frac{4}{5}\right)^{8-r} - \binom{8}{0} \times \left(\frac{1}{5}\right)^0 \times \left(\frac{4}{5}\right)^8 - \binom{8}{1} \times \left(\frac{1}{5}\right)^1 \times \left(\frac{4}{5}\right)^7$$

$$= 1 - \binom{8}{0} \times \left(\frac{1}{5}\right)^0 \times \left(\frac{4}{5}\right)^8 - \binom{8}{1} \times \left(\frac{1}{5}\right)^1 \times \left(\frac{4}{5}\right)^7$$

$$\approx 0.497 \,.$$

In the above calculations we have made use of the *Binomial Theorem*

$$(a + b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r} \,,$$

with

$$a = \frac{1}{5}, \qquad b = \frac{4}{5}, \quad \text{and} \quad n = 8 \,.$$

A simpler solution could have been given if we looked at the probability of $\overline{\mathcal{B}}$ (the complementary event to $\mathcal{B}$).