

Mathematics 1T (Algebra)

Summary of Week #10

• Probability–basic facts

1. Vocabulary:

- (a) **experiment** \equiv any situation where we know all the possibilities but we do not know which one will occur on any given occasion.
- (b) **outcome** \equiv each of the possibilities of the experiment.
- (c) **event** \equiv a group of outcomes considered together and given a neat verbal description.

OBS. This terminology is best understood in particular contexts. Observe the following examples:

experiment = “toss a coin”
event = “a tail faces upwards”
number of possible outcomes of the *experiment* = 2 (head or tail)
number of possible outcomes of the *event* = 1

experiment = “roll a dice once”
event = “a five faces upwards”
number of possible outcomes of the *experiment* = 6
number of possible outcomes of the *event* = 1

experiment = “select 3 people from a group of 5 men and 2 women”
event = “the group of people chosen contains 2 men and 1 woman”
number of possible outcomes of the *experiment* = $\binom{7}{3}$
number of possible outcomes of the *event* = $\binom{5}{2} \times \binom{2}{1}$

• **Fundamental Law of Probability**

Let \mathcal{E} be an experiment that has n possible outcomes, *all of which are equally likely to occur*. Let \mathcal{A} be an event that corresponds to r of the outcomes. Then

$$P(\mathcal{A}) = \frac{r}{n} \equiv \frac{\text{total number of outcomes (event)}}{\text{total number of outcomes (experiment)}}. \quad (1)$$

OBS. To find the numbers r and n we shall always have to count outcomes. This involves applying the *Principles of Counting*; also, a good understanding of *permutations* and *combinations* is essential for the remaining of the course.

- **Complementary events:** Sometimes we want to know the probability that a given event will not happen. An event opposite to the event of interest is called a *complementary* (or an *opposite*) event. For a given event, say, \mathcal{A} , its complementary is denoted by $\overline{\mathcal{A}}$.
- **Mutually exclusive events:** Two events \mathcal{A} and \mathcal{B} are said to be *mutually exclusive* if they cannot both happen at the same time. This concept can be extended in an obvious way to an arbitrary (but finite) number of events.
- **Properties of probabilities:**

1. $0 \leq P(\mathcal{A}) \leq 1$ for any event \mathcal{A} .
2. $P(\mathcal{A}) = 0$ if and only if \mathcal{A} is an *impossible event*, i.e. an event that cannot actually occur.
3. $P(\mathcal{A}) = 1$ if and only if \mathcal{A} is an event that is absolutely certain to occur.
4. For any event \mathcal{A} ,

$$P(\overline{\mathcal{A}}) = 1 - P(\mathcal{A}) \quad \text{or} \quad P(\mathcal{A}) = 1 - P(\overline{\mathcal{A}}). \quad (2)$$

5. If \mathcal{A}_1 and \mathcal{A}_2 are mutually exclusive events, then

$$P(\mathcal{A}_1 \cup \mathcal{A}_2) = P(\mathcal{A}_1) + P(\mathcal{A}_2). \quad (3)$$

Here, $\mathcal{A}_1 \cup \mathcal{A}_2$ is the mathematical notation for the event “ \mathcal{A}_1 or \mathcal{A}_2 ”.

6. If $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_p$ are mutually exclusive events, then

$$P(\mathcal{A}_1 \cup \mathcal{A}_2 \cup \dots \cup \mathcal{A}_p) = P(\mathcal{A}_1) + P(\mathcal{A}_2) + \dots + P(\mathcal{A}_p).$$

7. If $m, n, r, s \in \mathbb{N}$, an expression of the form

$$\frac{\binom{m}{r} \times \binom{n}{s}}{\binom{m+n}{r+s}}.$$

is said to be of **hypergeometric pattern**.

Example #1:

A bag contains two different types of apples: 6 Pink Lady (PK) and 4 McIntosh (MI). A man chooses 3 apples at random from the bag. What are

- (i). the probability that he gets 3 PK?
- (ii). the probability that he gets *precisely* 2 PK?
- (iii). the probability that he gets *at least* 1 PK and *at least* 1 MI?
- (iv). the probability that he gets *at least* 1 MI?

Solution

Before we attempt to solve problems like this, it is a good idea to label the events whose probabilities we are required to calculate. Thus, we introduce the following notation:

$$\begin{aligned}\mathcal{A}_1 &= \text{the 3 apples chosen are all PK,} & N_1 &= \binom{6}{3}, \\ \mathcal{A}_2 &= \text{the 3 apples chosen include precisely 2 PK,} & N_2 &= \binom{6}{2} \times \binom{4}{1}, \\ \mathcal{A}_3 &= \text{the 3 apples chosen include at least 1 PK and 1 MI,} \\ \mathcal{A}_4 &= \text{the 3 apples chosen include at least 1 MI,}\end{aligned}$$

where we have agreed to denote by N_1 , N_2 , N_3 , and N_4 , the total number of possible outcomes of the events \mathcal{A}_1 , \mathcal{A}_2 , \mathcal{A}_3 , and \mathcal{A}_4 , respectively. The numbers N_1 and N_2 recorded above are calculated with the help of the principles of counting studied previously; however, N_3 and N_4 are slightly more difficult to obtain, so this will be explained in detail below.

First, we notice that the total number of outcomes of the experiment (i.e. choosing 3 apples at random from the bag) is equal to

$$N = \binom{10}{3}.$$

According to the *Fundamental Law of Probability*, we have

$$P(\mathcal{A}_1) = \frac{N_1}{N} = \frac{\binom{6}{3}}{\binom{10}{3}} = \frac{1}{6},$$

and this result answers (i). The second part of the example is amenable to a similar argument. Without going into any further details,

$$P(\mathcal{A}_2) = \frac{N_2}{N} = \frac{\binom{6}{2} \times \binom{4}{1}}{\binom{10}{3}} = \frac{1}{2}.$$

(iii). For this question, it seems reasonable to break down \mathcal{A}_3 in two *mutually exclusive* events, \mathcal{B} and \mathcal{C} , defined as follows:

$$\begin{aligned}\mathcal{B} &= \text{the 3 apples chosen include precisely 2 PK and 1 MI,} & N_B &= \binom{6}{2} \times \binom{4}{1}, \\ \mathcal{C} &= \text{the 3 apples chosen include precisely 1 PK and 2 MI,} & N_C &= \binom{6}{1} \times \binom{4}{2}.\end{aligned}$$

The numbers N_B and N_C represent the total number of possible outcomes for \mathcal{B} and \mathcal{C} , respectively. Since

$$\mathcal{A}_3 = \mathcal{B} \cup \mathcal{C}, \tag{4}$$

and the two events on the right-hand side of (4) are mutually exclusive, we are in position to apply (3). Hence,

$$P(\mathcal{A}_3) = P(\mathcal{B} \cup \mathcal{C})$$

$$\begin{aligned}
&= P(\mathcal{B}) + P(\mathcal{C}) \\
&= \frac{N_B}{N} + \frac{N_C}{N} \\
&= \frac{\binom{6}{2} \times \binom{4}{1}}{\binom{10}{3}} + \frac{\binom{6}{1} \times \binom{4}{2}}{\binom{10}{3}} \\
&= \frac{1}{2} + \frac{3}{10} = \frac{4}{5}.
\end{aligned}$$

(iv). Finally, the last part is a typical question that requires the application of (2) since it is simpler to work out $P(\overline{\mathcal{A}}_4)$ rather than $P(\mathcal{A}_4)$ directly. Note that

$$\overline{\mathcal{A}}_4 = \text{“not 3 PK”} \equiv \overline{\mathcal{A}}_1.$$

Hence,

$$\begin{aligned}
P(\mathcal{A}_4) &= P(\overline{\mathcal{A}}_1) \\
&= 1 - P(\mathcal{A}_1) \\
&= 1 - \frac{1}{6} = \frac{5}{6}.
\end{aligned}$$

Conditional probability

- If \mathcal{A}_1 and \mathcal{A}_2 are two given events, the probability of \mathcal{A}_2 given \mathcal{A}_1 is a so-called *conditional probability*, and is denoted by

$$P(\mathcal{A}_2 | \mathcal{A}_1).$$

- For any two events \mathcal{A}_1 and \mathcal{A}_2 ,

$$P(\mathcal{A}_1 \cap \mathcal{A}_2) = P(\mathcal{A}_1)P(\mathcal{A}_2 | \mathcal{A}_1) \quad (5)$$

and

$$P(\mathcal{A}_1 \cap \mathcal{A}_2) = P(\mathcal{A}_2)P(\mathcal{A}_1 | \mathcal{A}_2). \quad (6)$$

- For any three events \mathcal{A}_1 , \mathcal{A}_2 and \mathcal{A}_3 ,

$$P(\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3) = P(\mathcal{A}_1)P(\mathcal{A}_2 | \mathcal{A}_1)P(\mathcal{A}_3 | \mathcal{A}_1 \cap \mathcal{A}_2). \quad (7)$$

This property can be generalised in an obvious way to any finite number of events.

Example #2:

A bag contains 6 silver coins and 4 gold. Two coins are drawn at random, one after the other, without replacement.

1. What is the probability of drawing first a silver and then a gold coin?
2. Assuming now that three coins are drawn at random, what is the probability of getting silver, then gold, and then silver again?

Solution

We start by letting \mathcal{A}_1 and \mathcal{A}_2 be the events

$$\begin{aligned}\mathcal{A}_1 &= \text{1st coin is silver}, & N_1 &= 6, \\ \mathcal{A}_2 &= \text{2nd coin is gold}, & N_2 &= 4,\end{aligned}$$

where N_1 is the total number of outcomes for \mathcal{A}_1 , and N_2 is the total number of outcomes of the event \mathcal{A}_2 .

Next, recognise that for the first part of the problem, we need to calculate

$$P(\mathcal{A}_1 \cap \mathcal{A}_2).$$

To this end, we employ (5),

$$P(\mathcal{A}_1 \cap \mathcal{A}_2) = P(\mathcal{A}_1)P(\mathcal{A}_2 | \mathcal{A}_1), \quad (8)$$

which leaves us with calculating the two probabilities on the RHS of (8). At the first draw, the total number of outcomes of the experiment is equal to 10 (initially, there are 10 coins in the bag), so that

$$P(\mathcal{A}_1) = \frac{6}{10}. \quad (9)$$

The total number of coins left after \mathcal{A}_1 has occurred is $10 - 1 = 9$, and thus

$$P(\mathcal{A}_2 | \mathcal{A}_1) = \frac{4}{9}. \quad (10)$$

Using (9) and (10) in (8), we obtain

$$\text{answer} = \frac{6}{10} \times \frac{4}{9} = \frac{4}{15}.$$

The second part of the problem is handled similarly, except that this time we need to introduce an extra event,

$$\mathcal{A}_3 = \text{3rd coin is silver}, \quad N_3 = 5,$$

and notice that this second question involves finding

$$P(\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3).$$

From (7) the answer is going to be equal to

$$P(\mathcal{A}_1)P(\mathcal{A}_2 | \mathcal{A}_1)P(\mathcal{A}_3 | \mathcal{A}_1 \cap \mathcal{A}_2). \quad (11)$$

The first two probabilities in this product have already been calculated. To find the third one we must take into account that there are only $10 - 2 = 8$ coins left after \mathcal{A}_1 and \mathcal{A}_2 have already occurred. Also, the occurrence of \mathcal{A}_1 means that there is one less silver coins left in the bag by the time we draw the third coin (that is why $N_3 = 6 - 1 = 5$). In conclusion,

$$P(\mathcal{A}_3 | \mathcal{A}_1 \cap \mathcal{A}_2) = \frac{5}{8}. \quad (12)$$

The final answer will follow from (11) and (12),

$$\text{answer} = \frac{6}{10} \times \frac{4}{9} \times \frac{5}{8}.$$

Independent events

- If the likelihood of an event \mathcal{A}_2 occurring is unaffected by whether an event \mathcal{A}_1 has or has not occurred, then we say that \mathcal{A}_1 and \mathcal{A}_2 are *independent events*.
- When \mathcal{A}_1 and \mathcal{A}_2 are independent events,

$$P(\mathcal{A}_1 \cap \mathcal{A}_2) = P(\mathcal{A}_1)P(\mathcal{A}_2). \quad (13)$$

The extension to an arbitrary (but finite) number of independent events is straightforward.

Example #3:

A dice is thrown repeatedly. What is the probability that a six occurs for the first time on the third throw.

Solution

Let \mathcal{A}_1 , \mathcal{A}_2 , and \mathcal{A}_3 be the following events:

$$\begin{aligned} \mathcal{A}_1 &= \text{not } 6 \text{ on the 1st throw,} \\ \mathcal{A}_2 &= \text{not } 6 \text{ on the 2nd throw,} \\ \mathcal{A}_3 &= 6 \text{ on the 3rd throw.} \end{aligned}$$

Note that the required probability is

$$P(\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3).$$

Clearly, the events defined above are independent of each other, so that according to the counterpart of (13) written for these three independent events,

$$P(\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3) = P(\mathcal{A}_1)P(\mathcal{A}_2)P(\mathcal{A}_3). \quad (14)$$

All we have to do now is evaluate the individual probabilities that appear on the RHS of (14). They are all easily found with the help of the *Fundamental Law of Probability*,

$$P(\mathcal{A}_1) = \frac{5}{6}, \quad P(\mathcal{A}_2) = \frac{5}{6}, \quad P(\mathcal{A}_3) = \frac{1}{6}. \quad (15)$$

Finally, from (14) and (15), we find

$$P(\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3) = \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} = \frac{25}{216}.$$