CFM2104: Applied Mathematics

Chapter 7: Linear Momentum

School of Computing & Engineering

March 2020/21

University of
HUDDERSFIELD

Learning Objectives:

(This is the **second** set of lecture notes on linear momentum.)

- explore an important application of the **impulse-momentum theorem**;

- introduce a general strategy for deriving differential equations of motion in situations in which the mass is variable.

Learning Outcome:

By the end of this week you should be able to set up the relevant equations of motion for different scenarios involving variable mass (e.g., Q.12, 13, 14, 15 from *Problem Set # 7*). You are also expected to solve these differential equations by using what you learned in the *Calculus* module.

5.13 Motivation

In all of the problems considered so far we have assumed that the mass of the objects involved has remained constant. However, this is not always true:

- the mass of a rocket reduces as its engine burns fuel
- the mass of a raindrop increases as it passes through a cloud

The equations of motion in these cases are best derived from first principles by using the **impulse-momentum theorem** (covered in the first set of notes)

$$\Delta \boldsymbol{p} = \boldsymbol{F}_{\text{net}} \Delta t$$

or

$$\boxed{\boldsymbol{p}(t + \Delta t) - \boldsymbol{p}(t) = \boldsymbol{F}_{\text{net}} \Delta t}$$

($\boldsymbol{F}_{\text{net}}$ is assumed to be constant in the interval of time Δt)

5.14 Example 1

A body is falling under gravity. It picks up matter as it falls. At time t the body has mass m and speed v. Assuming that the matter picked was stationary before it coalesced with the body, derive the differential equation for the motion of the body.

Initially (i.e., at time t) we have two elements, the body of mass m, moving with speed v, and the additional matter of mass δm, which is not moving:

After an interval δt (i.e., at time $t + \delta t$) we have a body of mass $m + \delta m$ moving with speed $v + \delta v$:
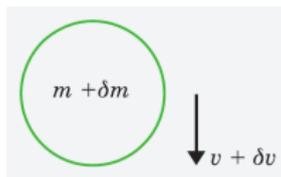
5.14 Example 1 (cont'd)

The external force acting on the system is the weight $(m + \delta m)g$

Use the impulse-momentum principle: **change in momentum = impulse**

$$\left[(m + \delta m)(v + \delta v)\right] - \left[mv + \delta m(0)\right] = (m + \delta m)g\delta t$$

Expand:

$$mv + m\delta v + v\delta m + (\delta m)(\delta v) - mv = mg\delta t + g(\delta m)(\delta t)$$

Divide this expression by δt:

$$\implies \quad m\frac{\delta v}{\delta t} + v\frac{\delta m}{\delta t} + \frac{(\delta m)(\delta v)}{\delta t} = mg + g\delta m$$

Take the limit as $\delta t \to 0$:

$$m\frac{dv}{dt} + v\frac{dm}{dt} = mg \qquad \text{or} \qquad \frac{d}{dt}(mv) = mg$$

5.14 Example 2

A body is falling under gravity. It picks up matter as it falls. At time t the body has mass m and speed v. Assuming that the matter picked up was falling with speed u before it coalesced with the body, derive the differential equation for the motion of the body.

Initially (i.e., at time t) we have two elements, the body of mass m, moving with speed v, and the additional matter of mass δm, moving with speed u:

After an interval δt (i.e., at time $t + \delta t$) we have a body of mass $m + \delta m$ moving with speed $v + \delta v$:
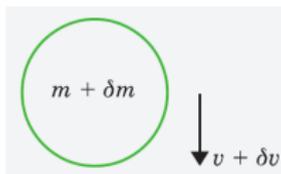
5.14 Example 2 (cont'd)

The external force acting on the system is the weight $(m + \delta m)g$

Use the impulse-momentum principle: **change in momentum = impulse**

$$\left[(m + \delta m)(v + \delta v)\right] - \left[mv + \delta m(u)\right] = (m + \delta m)g\delta t$$

Expand:

$$mv + m\delta v + v\delta m + (\delta m)(\delta v) - mv - u\delta m = mg\delta t + g(\delta m)(\delta t)$$

Divide this expression by δt:

$$\implies \quad m\frac{\delta v}{\delta t} + v\frac{\delta m}{\delta t} + \frac{(\delta m)(\delta v)}{\delta t} - u\frac{\delta m}{\delta t} = mg + g\delta m$$

Take the limit as $\delta t \to 0$:

$$m\frac{dv}{dt} + v\frac{dm}{dt} - u\frac{dm}{dt} = mg \qquad \text{or} \qquad \boxed{m\frac{dv}{dt} + (v - u)\frac{dm}{dt} = mg}$$
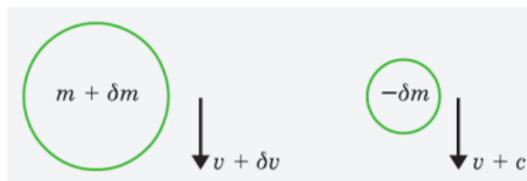
5.14 Example 3

A spaceship is moving in deep space with no external forces acting on it. At time t the spaceship has total mass m and is moving with velocity v. The spaceship reduces its speed by ejecting fuel from its front end with a speed c relative to itself and in the same direction as its own motion. Show that

$$\frac{dv}{dm} = \frac{c}{m}$$

Initially
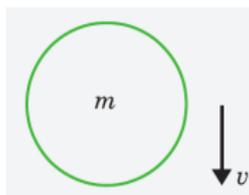
After an interval δt:

5.14 Example 3 (cont'd)

The are NO external force acting on the system (hence zero impulse)

Use the impulse-momentum principle: **change in momentum = impulse=0**

$$\left[(m + \delta m)(v + \delta v) + (-\delta m)(v + c)\right] - mv = 0$$

Expand:

$$mv + m\delta v + v\delta m + (\delta m)(\delta v) - v\delta m - c\delta m - mv = 0$$

$$m\delta v + (\delta m)(\delta v) - c\delta m = 0$$

Divide this expression by δm:

$$\implies \quad m\frac{\delta v}{\delta m} + \delta v - c = 0$$

Take the limit as $\delta t \to 0$ (hence $\delta m \to 0$ and $\delta v \to 0$):

$$m\frac{dv}{dm} - c = 0 \qquad \text{or} \qquad \boxed{\frac{dv}{dm} = \frac{c}{m}}$$
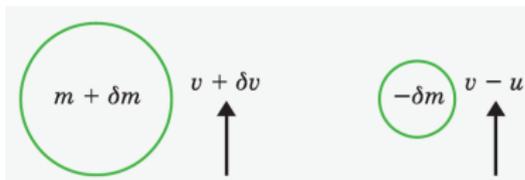
5.14 Example 4

A rocket is launched vertically upwards under gravity from rest at time $t = 0$. The rocket propels itself upwards by ejecting fuel vertically downwards at a constant speed u relative to the rocket. The initial mass of the rocket, including fuel, is M. At time t, before all the fuel has been used up, the mass of the rocket (including fuel) is $M(1 - kt)$ and the speed of the rocket is v. Air resistance can be ignored. Show that

$$\frac{dv}{dt} = \frac{ku}{1 - kt} - g$$

Initially
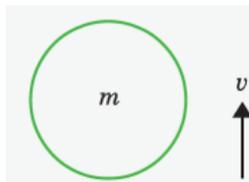
After an interval δt:

5.14 Example 4 (cont'd)

Use the impulse-momentum principle: **change in momentum = impulse**

$$\left[(m + \delta m)(v + \delta v) + (-\delta m)(v - u)\right] - mv = -mg\delta t$$

Expand:

$$mv + m\delta v + v\delta m + (\delta m)(\delta v) - v\delta m + u\delta m - mv = -mg\delta t$$

Divide this expression by δt:

$$\implies \quad m\frac{\delta v}{\delta t} + u\frac{\delta m}{\delta t} + \frac{(\delta m)(\delta v)}{\delta t} = -mg$$

Take the limit as $\delta t \to 0$:

$$\boxed{m\frac{dv}{dt} + u\frac{dm}{dt} = -mg} \qquad \text{since } m = M(1 - kt) \implies$$

$$\implies M(1 - kt)\frac{dv}{dt} + u(-kM) = -M(1 - kt)g \implies \boxed{\frac{dv}{dt} = \frac{ku}{1 - kt} - g}$$

5.14 Example 5

A raindrop falls through a stationary cloud. When the raindrop has fallen a distance x it has mass m and speed v. The mass increases uniformly by accretion so that $m = M(1 + kx)$, where M and k are known constants. Air resistance may be asumed negligible. Given that $v = 0$ when $x = 0$, find the speed of the raindrop when $x = 1/k$.

The diagrams are the same as in *Example 1.*
The resulting **equation is** also **identical,**

$$m\frac{dv}{dt} + v\frac{dm}{dt} = mg$$

Using the chain rule:

$$m\frac{dv}{dx}\frac{dx}{dt} + v\frac{dm}{dx}\frac{dx}{dt} = mg$$

Since $v = \dfrac{dx}{dt} \implies mv\dfrac{dv}{dx} + v^2\dfrac{dm}{dx} = mg \implies \dfrac{1}{2}m\dfrac{d}{dx}(v^2) + v^2(kM) = mg$

5.14 Example 5 (cont'd)

Linear ODE in v^2. Let's introduce the notation $Y = v^2$. The ODE becomes

$$\boxed{\frac{dY}{dx} + \left(\frac{2k}{1+kx}\right)Y = 2g}$$ integration factor (I.F.) $= (1+kx)^2$

Multiplying the ODE by the I.F. we get

$$\frac{d}{dx}[Y(1+kx)^2] = 2g(1+kx)^2 \implies v^2(1+kx)^2 = \frac{2g}{3k}(1+kx)^3 + C$$

Use the initial conditions to find the integration constant

$$x = 0, \ v = 0 \implies 0 = \frac{2g}{3k} + C$$

We finally get

$$v^2 = \frac{2g}{3k}(1+kx) - \frac{2g}{3k(1+kx)^2}$$

Substituting $x = 1/k$ in this expression we find $v = (7g/6k)^{1/2}$.

5.14 Points to remember

- You should approach each problem from **first principles** (following the model of the previous examples).

- The **impulse-momentum theorem** states that the change of linear momentum of the system in the time interval δt (i.e., from t to $t + \delta t$) is equal to the impulse of the external forces acting on the system in that time interval.

- **Always** use $m + \delta m$ for the mass at the end of the interval, even if the moving object is actually losing mass.

- Pay close attention to the arrows in the diagrams included with the previous examples (they all point in the *same* direction)