University of Huddersfield
School of Computing & Engineering

Problem Set # 7   **Polar Coordinates**
                                                              Central forces

1. Draw on the xy-plane the points with polar coordinates:

   (i). $r = 1$, $\theta = \pi/3$;   (iv). $r = 2$, $\theta = -5\pi/6$;

   (ii). $r = 1$, $\theta = -3\pi$;   (v). $r = 0$, $\theta = \pi/2$.

   For each point, calculate its x- and y-coordinates.
   For each point, draw on your diagram the associated unit vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$.

2. Draw on the xy-plane the points with Cartesian coordinates:

   (i). $x = 1$, $y = -\sqrt{3}$;   (iv). $x = -1$, $y = \sqrt{3}$;

   (ii). $x = 1$, $y = 4$;   (v). $x = -4$, $y = -1$.

   For each point, calculate its polar coordinates.
   For each point, draw on your diagram the associated unit vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$.

3. For each subquestion below: (a) find the eccentricity and an equation of the directrix of the conic; (b) identify the conic; (c) sketch the curve.

   (i). $r = \dfrac{8}{6 + 2\sin\theta}$;   (iv). $r = \dfrac{1}{3 - 2\cos\theta}$;

   (ii). $r = \dfrac{10}{4 + 6\cos\theta}$;   (v). $r = \dfrac{1}{1 - \sin\theta}$;

   (iii). $r = \dfrac{5}{2 + 2\cos\theta}$;   (vi). $r = -\dfrac{6}{\sin\theta - 2}$.

4. (a) Show that the polar equation of an ellipse with one focus at the pole and major axis lying along the polar axis is given by

$$r = \frac{a(1 - e^2)}{1 - e\cos\theta},$$

   where e is the eccentricity of the ellipse and $2a$ is the length of the major axis.

   (b) A planet revolves around the Sun in an elliptical orbit as above, with the Sun at one focus. Use the results from part (a) to show that the perihelion distance (minimum distance from the planet to the Sun) is $a(1 - e)$.
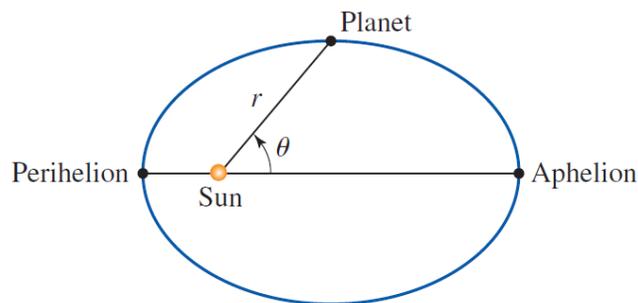
5. Find the radial and transverse components of the acceleration of a particle moving at unit speed along the curve $r = e^\theta$. Express your answers in terms of the angle θ.

6. A particle P moves in a plane. At time t seconds its polar coordinates (r, θ) relative to a fixed origin are given by

$$r = 2at, \qquad \theta = t^2 - t,$$

where a is a positive constant. Find the speed of P when $t = 2$.

7. Let $a > 0$ be a given constant. A particle moves along the curve with polar equation $r\theta = a$ (with $\theta \geq \pi/4$) in such a way that the transverse component of the acceleration is always zero. Show that the radial component of the acceleration is inversely proportional to r^3.

8. If a particle moves along the polar curve $r = \theta$ under the influence of a central force attracting it to the origin, show the magnitude (a) of the acceleration is

$$a = \frac{(2 + r^2)v^2}{r(1 + r^2)},$$

where v is the speed of the particle.

9. An object moves along the polar curve $r = \theta^{-2}$ under the influence of a central force directed towards the origin. If the speed of the object is v_0 at the moment when $\theta = 1$, show that the magnitude of the acceleration (a) of the object at any point on its path is

$$a = \frac{v_0^2}{5}\left(\frac{2}{r^2} + \frac{1}{r^3}\right).$$

10. One end of an elastic string of natural length a and elastic constant mg/a is attached to a fixed point O on a smooth horizontal table and the other to a particle P of mass m on the table. The particle is projected with speed v in a direction perpendicular to the string when the length of the string is $3a$. Given that the maximum length of the string in the subsequent motion is $5a$, show that:

$$v^2 = \frac{75}{4}ga.$$

11. A particle moves round a curve with polar equation $r = 2a \cos \theta \ (-\pi/2 \le \theta \le \pi/2)$, where $a > 0$ is a constant, under the action of a central force of attraction directed towards O, the pole.

   (a) Explain why the curve mentioned above is in fact a circle.
   (b) Show that $\dot{r} = (-2ah/r^2) \sin \theta$, where $h = r^2 \dot{\theta}$.
   (c) Show also that the magnitude of the force is proportional to r^{-5}.

12. At time t the polar coordinates of a particle of unit mass moving in a plane are (r, θ). The only force acting on the particle is

$$\boldsymbol{F} = \frac{\mu}{r^3} \boldsymbol{e}_r \,,$$

   where μ is a constant and $\boldsymbol{e}_r$ is the usual unit vector along the radial direction. Show that:

   $(a) \qquad \dfrac{d\theta}{dt} = \dfrac{h}{r^2} \,;$

   $(b) \qquad \dfrac{d^2 r}{dt^2} = \dfrac{\mu + h^2}{r^3} \,, \qquad (h = \text{constant}) \,.$

13. A particle P, of unit mass, moves on a smooth horizontal plane under the action of a force of magnitude
$$w^2 r + \frac{w^2 a^3}{r^2} \,, \qquad (w = \text{constant})$$
directed towards a fixed point O of the plane, where $OP = r$. The particle is projected from a point, a distance a from O, with a horizontal speed $4aw/\sqrt{3}$ in a direction perpendicular to OP.

   (a) Show that in the subsequent motion

$$\dot{r}^2 = \frac{w^2 (r - a)(2a - r)(3r^2 + 9ar + 8a^2)}{3r^2} \,.$$

   (b) Deduce that $a \le r \le 2a$.

14. A particle P moves in a path with polar equation

$$r = \frac{2a}{2 + \cos \theta}$$

   with respect to a pole O and initial line OA. At any time t during the motion it is given that $r^2 \dot{\theta} = h$ (constant). Show that the acceleration of P is directed towards O and that its magnitude is inversely proportional to r^2.