## University of Huddersfield
### School of Computing & Engineering

# Problem Set # 5 <span style="float:right">**Kinematics**</span>

1. Find $\boldsymbol{r}(t)$ satisfying the given conditions:

   (i).   $\dot{\boldsymbol{r}}(t) = 2\boldsymbol{i} + 4t\boldsymbol{j} - 6t^2\boldsymbol{k}$,     $\boldsymbol{r}(0) = \boldsymbol{i} + \boldsymbol{k}$;

   (ii).   $\dot{\boldsymbol{r}}(t) = \sqrt{t+1}\boldsymbol{i} + \dfrac{t}{t^2+1}\boldsymbol{j} + \dfrac{1}{t}\boldsymbol{k}$,     $\boldsymbol{r}(3) = \boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$;

   (iii).   $\ddot{\boldsymbol{r}}(t) = \sqrt{t}\boldsymbol{i} + \sec^2 t\boldsymbol{j} + \mathrm{e}^t\boldsymbol{k}$,     $\dot{\boldsymbol{r}}(0) = \boldsymbol{i} + \boldsymbol{k}$,   $\boldsymbol{r}(0) = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$;

   (iv).   $\ddot{\boldsymbol{r}}(t) = 3\cos 2t\boldsymbol{i} + 4\sin 2t\boldsymbol{j} + \boldsymbol{k}$,     $\dot{\boldsymbol{r}}(0) = \boldsymbol{i} + 2\boldsymbol{j}$,   $\boldsymbol{r}(0) = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$.

2. Show that if $\boldsymbol{r}(t) = \mathrm{e}^{kt}\boldsymbol{i} + \mathrm{e}^{-kt}\boldsymbol{j}$ ($k \in \mathbb{R}$), then the vectors $\boldsymbol{r}$ and $\ddot{\boldsymbol{r}}$ are parallel.

3. Suppose that the the vectors $\boldsymbol{u}(t)$ and $\ddot{\boldsymbol{u}}(t)$ are parallel for all $t$. Prove that $\boldsymbol{u} \wedge \dot{\boldsymbol{u}}$ is a constant vector.

4. For a particle moving in 3D-space with constant acceleration $\boldsymbol{a}$, show that
$$v^2 = v_0^2 + 2\boldsymbol{a} \cdot \boldsymbol{s},$$
where $\boldsymbol{s} = \boldsymbol{r}(t) - \boldsymbol{r}_0$ (the displacement of the particle at time $t$ relative to where it started). Here, $v = |\boldsymbol{v}|$ (the speed at time $t$) and $v_0 = |\boldsymbol{v}_0|$ (the initial speed).

5. Two buses $A$ and $B$ move along a straight road. $B$ leaves the depot at $10:00$ am travelling at constant speed of $50$ Km/h. Bus $A$ leaves the depot at $10:15$ am and travels in the same direction at constant speed of $60$ Km/h. Find when and where $A$ overtakes $B$.

6. An object moves to the right along the curve $y = x^2$ with constant speed $v = 5$. Find the velocity and acceleration of the object when it is at the point $(1,1)$.

7. The position vector of a particle that moves in 3D-space is given by
$$\boldsymbol{r}(t) = 3u\boldsymbol{i} + 3u^2\boldsymbol{j} + 2u^3\boldsymbol{k},$$
where $u = u(t)$ is a differentiable function. Knowing that the particle moves in the direction corresponding to increasing $u$ (i.e., $\dot{u} > 0$) and with constant speed of $6$, find the velocity and acceleration of the particle when it is at the point $(3,3,2)$.
[**Hint:** use the chain rule in calculating $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$.]

8. An object moves along the curve described by the parametric equations
$$x = x(t), \qquad y = x^2, \qquad z = x^3,$$
with constant vertical speed $dz/dt = 3$. Find the velocity and acceleration of the object when it is at the point $(2,4,8)$.

9. If at all times $t$ the position and velocity vectors of a moving particle satisfy $\boldsymbol{v}(t) = 2\boldsymbol{r}(t)$, and if $\boldsymbol{r}(0) = \boldsymbol{r}_0$ (given vector), find $\boldsymbol{r}(t)$ and the acceleration $\boldsymbol{a}(t)$. What is the path of the motion?

10. Verify that $\boldsymbol{r}(t) = \boldsymbol{r}_0 \cos \omega t + (\boldsymbol{v}_0/\omega) \sin \omega t$ satisfies the IVP:

$$\frac{d^2\boldsymbol{r}}{dt^2} = -\omega^2 \boldsymbol{r}, \qquad \dot{\boldsymbol{r}}(0) = \boldsymbol{v}_0, \quad \boldsymbol{r}(0) = \boldsymbol{r}_0.$$

What is the path described by $\boldsymbol{r}(t)$? What is the path if $\boldsymbol{r}_0$ is perpendicular to $\boldsymbol{v}_0$?

11. A particle moves along the circle $x^2 + y^2 = 25$ at constant speed, making one revolution in $2$ seconds. Find its acceleration when it is at the point $(3, 4)$.

12. The position vector $\boldsymbol{r} = \boldsymbol{r}(t)$ of a moving particle $P$ satisfies the IVP:

$$\frac{d\boldsymbol{r}}{dt} = 2\boldsymbol{i} \wedge \boldsymbol{r}, \qquad \boldsymbol{r}(0) = \boldsymbol{i} + 3\boldsymbol{j}.$$

Find $\boldsymbol{r}(t)$ and describe the motion of $P$.

[**Hint:** write the equation in components and then solve the resulting ODEs.]

13. At what point during its flight is the speed of a projectile minimum? A particle is projected from a point $O$ on a horizontal plane with an angle of projection $\alpha$. Show that the ratio of the greatest speed to the least speed during the flight is $1/\cos \alpha$.

14. A tile slides down a roof inclined at $20°$ to the horizontal starting $3$ m from the edge of the roof. Assuming that the roof is smooth find the horizontal distance from the edge of the roof that the tile hits the ground if the edge of the roof is $8$ m above the ground.

15. A particle is projected from a point $O$ with an initial speed of $30$ m/s to pass through a point which is $40$ m from $O$ horizontally and $10$ m above $O$. Show that there are two angles of projection for which this is possible. If the angles are $\alpha$ and $\beta$, show that $\tan(\alpha + \beta) = -4$.

16. A particle is projected with an initial speed of $60$ m/s towards a wall which is $100$ m horizontally from the point of projection and $20$ m high. Find the least angle of projection for which the particle will pass over the wall.

17. A particle is projected from a point $O$ with an initial velocity of $60$ m/s at an angle of $30°$ to the horizontal. At the same instant a second particle is projected in the opposite direction with initial speed $50$ m/s from a point level with $O$ and $100$ m from $O$. Find the angle of projection of the second particle if they collide. Find also the time at which the collision takes place.

18. A particle is projected from a point $O$ with an initial velocity of $20$ m/s and at an angle of $\tan^{-1}(3/4)$ to the horizontal. Two seconds later a second particle is projected from $O$ and it collides with the first particle $1$ second after leaving $O$. Find the initial velocity of the second particle (take $g = 10$ m/s$^2$).

19. (a) Let $\boldsymbol{r} = \boldsymbol{r}(t)$ be a differentiable vector function of $t$ and let $r = |\boldsymbol{r}|$. Show that

$$\boldsymbol{r} \cdot \frac{d\boldsymbol{r}}{dt} = r \frac{dr}{dt} \, .$$

(b) Hence, or otherwise, show that

$$\frac{d}{dt}\left(\frac{\boldsymbol{r}}{r}\right) = \frac{1}{r^3}\left[\left(\boldsymbol{r} \wedge \frac{d\boldsymbol{r}}{dt}\right) \wedge \boldsymbol{r}\right] \, .$$

(c) Show also that

$$\frac{d}{dt}\left(\frac{\boldsymbol{r}}{r}\right) = \frac{\dot{\boldsymbol{r}}}{r} - \left(\frac{\boldsymbol{r} \cdot \dot{\boldsymbol{r}}}{r^3}\right)\boldsymbol{r} \, .$$

[**Note:** this is an optional question (i.e., not examinable).]

20. Let $a_1$, $b_1$, etc be real numbers. Show that the curve $C$ described by the position vector

$$\boldsymbol{r}(t) = (a_1 t^2 + b_1 t + c)\boldsymbol{i} + (a_2 t^2 + b_2 t + c)\boldsymbol{j} + (a_3 t^2 + b_3 t + c)\boldsymbol{k}$$

lies in a plane.

[**Note:** this is an optional question (i.e., not examinable).]