

Problem Set # 1

Vectors

- If $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, evaluate $|(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})|$.
 - If $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, find the magnitude of the vector $(2\mathbf{a} + \mathbf{b}) \wedge (\mathbf{a} - 2\mathbf{b})$.
- In which of the following cases are the vectors \mathbf{a} and \mathbf{b} orthogonal
 - $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$?
 - $\mathbf{a} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$?
- For what value of $\lambda \in \mathbb{R}$ are the vectors \mathbf{a} and \mathbf{b} parallel
 - $\mathbf{a} = 3\mathbf{i} + \lambda\mathbf{j}$, $\mathbf{b} = 9\mathbf{i} - 12\mathbf{j}$?
 - $\mathbf{a} = \lambda\mathbf{i} - 8\mathbf{j}$, $\mathbf{b} = 7\mathbf{i} - 10\mathbf{j}$?
 - $\mathbf{a} = 2\mathbf{i} + \lambda\mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + \lambda\mathbf{k}$?
- Find the value of $\alpha \in \mathbb{R}$ for which the vectors $\mathbf{a} = -\mathbf{i} + (\alpha - 3)\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = \alpha\mathbf{i} + \mathbf{j} - 4\mathbf{k}$, $\mathbf{c} = 2\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}$ are coplanar.
- A force \mathbf{R} acts on a particle, where $\mathbf{R} = (7\mathbf{i} + 16\mathbf{j})$ N. Calculate:
 - the magnitude of \mathbf{R} (2dp);
 - the angle between the line of action of \mathbf{R} and \mathbf{i} , giving your answer to the nearest degree;
 - The force \mathbf{R} is the resultant of two forces \mathbf{P} and \mathbf{Q} . The line of action of \mathbf{P} is parallel to the vector $\mathbf{i} + 4\mathbf{j}$, and the line of action of \mathbf{Q} is parallel to the vector $\mathbf{i} + \mathbf{j}$. Determine the forces \mathbf{P} and \mathbf{Q} expressing each in terms of \mathbf{i} and \mathbf{j} .
- The force \mathbf{R} is given by the vector $6\mathbf{i} + 2\mathbf{j}$, where the units of force are newtons. This force is the resultant of a force \mathbf{P} parallel to \mathbf{i} and a force \mathbf{Q} parallel to $\mathbf{i} + \mathbf{j}$. Find the magnitudes of \mathbf{P} and \mathbf{Q} .
- (The law of cosines) In a triangle ABC ,

$$c^2 = a^2 + b^2 - 2ab \cos C,$$

where $a = BC$, $b = AC$, $c = AB$ and C stands for $\angle ACB$.

[Hint: if $\mathbf{a} = \overrightarrow{BC}$, $\mathbf{b} = \overrightarrow{AC}$, $\mathbf{c} = \overrightarrow{BA}$, then $\mathbf{c} = \mathbf{a} - \mathbf{b}$; use $\mathbf{c} \cdot \mathbf{c} = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$, etc.]

8. (*The law of sines*) In the context of the previous question show that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

where a, b, c are the lengths of the sides of the triangles and A, B, C are the opposite angles.

[**Hint:** Clearly, $\mathbf{b} + \mathbf{c} - \mathbf{a} = \mathbf{0}$; take the vector product of both sides with \mathbf{a} and \mathbf{b} , respectively, etc.]

9. (optional) If \mathbf{u} and \mathbf{v} are arbitrary vectors, then show that

$$(i). \quad |\mathbf{u} - \mathbf{v}|^2 + |\mathbf{u} + \mathbf{v}|^2 = 2(|\mathbf{u}|^2 + |\mathbf{v}|^2);$$

$$(ii). \quad \mathbf{u} \cdot \mathbf{v} = \frac{1}{4}(|\mathbf{u} + \mathbf{v}|^2 - |\mathbf{u} - \mathbf{v}|^2).$$

10. If vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are perpendicular, show that \mathbf{a} and \mathbf{b} are of equal magnitude.

11. (optional) If the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are represented by the sides $\overrightarrow{BC}, \overrightarrow{CA}, \overrightarrow{AB}$ of a triangle ABC , show that $\mathbf{b} \wedge \mathbf{c} = \mathbf{c} \wedge \mathbf{a} = \mathbf{a} \wedge \mathbf{b}$.

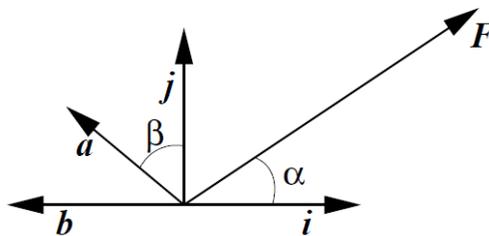
12. (optional) If the vectors $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are the position vectors of the vertices of triangle PQR , show that

$$\text{area}(PQR) = \frac{1}{2}|\mathbf{p} \wedge \mathbf{q} + \mathbf{q} \wedge \mathbf{r} + \mathbf{r} \wedge \mathbf{p}|.$$

[**Hint:** Do you know any relation between vector products and areas?]

13. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are arbitrary vectors, verify that $(\mathbf{c} - \mathbf{a}) \wedge (\mathbf{b} - \mathbf{a}) = (\mathbf{b} - \mathbf{c}) \wedge (\mathbf{a} - \mathbf{c})$.

14. Given that \mathbf{a}, \mathbf{b} are both *unit vectors*, and that the magnitude of the force \mathbf{F} is 10:
 (i) find the components of \mathbf{F} in the directions of \mathbf{i} and \mathbf{j} , respectively;
 (ii) find the components of \mathbf{F} in the direction of \mathbf{a} and \mathbf{b} (see the sketch included below).



15. (optional) The movements of an armoured tank in a desert are plotted at five minute intervals by a spy satellite. These are the four observed stages of its journey:

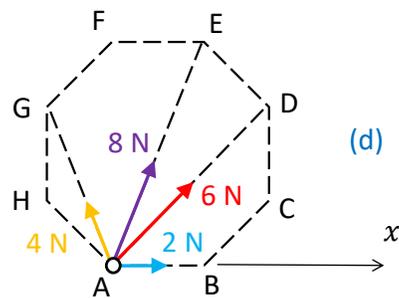
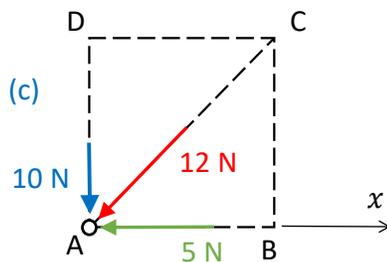
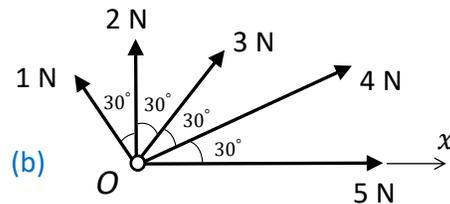
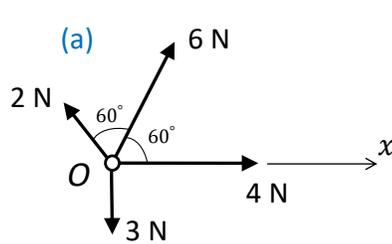
- First stage: 1.3 km, due north;
- Second stage: 1.1 km, due north-east;

- Third stage: 2.0 km on a bearing of 125° ;
- Fourth stage: 0.9 km on a bearing of 102° .

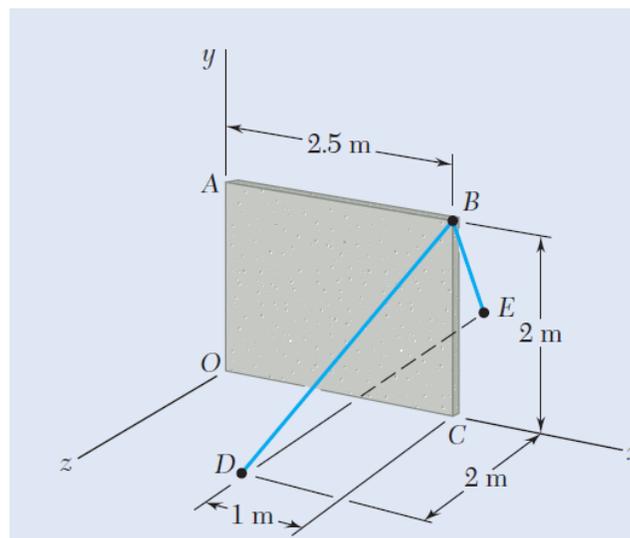
At the end of the fourth stage, what is the distance and bearing from its original position?

[Hint: Draw a big sketch; a *bearing* is an angle measured clockwise from the north direction.]

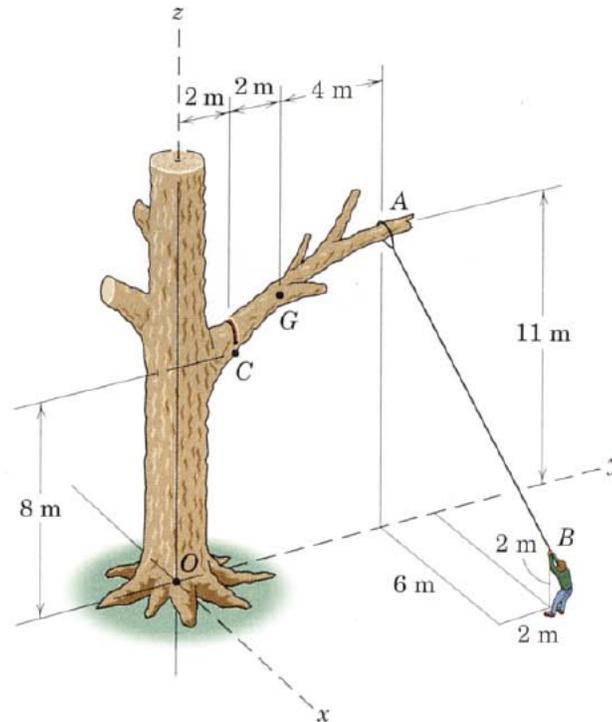
16. In each question from (a) to (d) shown below, find the magnitude of the resultant of the given vectors and work out the angle between the resultant and the direction of the positive x -axis; $ABCD$ is a square and $ABCDEFGH$ is a regular octagon.



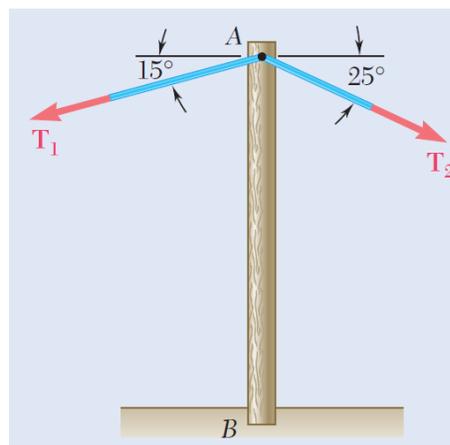
17. A precast wall section is temporarily held by two cables as shown in the sketch. Knowing that the tension in cable BD is 900 N, determine the vector moment about point O of the force exerted by the cable at B .



18. In an attempt to pull down a nearly sawn-through branch, the tree surgeon exerts a 400-N pull on the line which is looped around the branch at A (see attached sketch). Determine the vector moment about point C of the force exerted on the branch and calculate the magnitude of this moment.



19. A telephone cable is clamped at A to the pole AB as seen in the sketch below. Knowing that the tension $T_1 = 800$ lb, determine the required tension T_2 if the resultant force \mathbf{R} of the forces exerted by the cable at A is to be vertical. Find the magnitude of \mathbf{R} .



Note: Optional questions are not examinable.