University of Huddersfield
School of Computing and Engineering

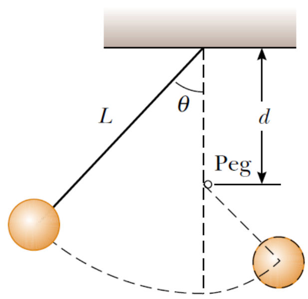
CFM2104 Applied Mathematics

Coursework Assignment **Submission deadline: 27 March 2019**

This assessed coursework should be submitted via Brightspace. Please show all your workings, add explanations were possible and make sure that the document is reasonably tidy and readable. Marks are assigned for all the following aspects: Methodology, clarity of exposition, correctness of results. Detailed marking criteria can be consulted on Brightspace.

---

1. (a) A pendulum comprising a string of length L and a sphere swings in the vertical plane. The string hits a peg located a distance d below the point of suspension (as seen in the sketch below).



   i. Show that if the sphere is released from a height below that of the peg, it will return to its height after striking the peg.

   ii. Show that if the pendulum is released from the horizontal position ($\theta = 90°$) and is to swing in a complete circle centred on the peg, then $d \geq 3L/5$.

   iii. Assuming that $d = L/2$ and the sphere is released as in the previous question, show that the maximum height reached by the pendulum above the lowest position (before the string becomes slack) is $5L/6$.

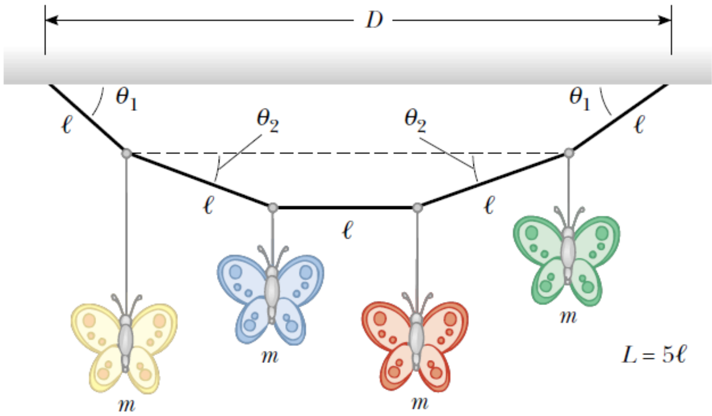
[15 marks]

(b) A particle A is thrown from horizontal ground from the origin of unit vectors $\mathbf{i}$ and $\mathbf{j}$, $\mathbf{i}$ being horizontal and $\mathbf{j}$ vertical, with velocity $2v_0$ at an angle α to $\mathbf{i}$. A particle B is thrown at the same instant with velocity v_0 at an angle *below the horizontal* from point $h\mathbf{j}$ to hit A in flight. Show that if $v_0^2 \sin\alpha$ is less than $hg/4R$, where $R = 2\sin\alpha + \sqrt{1 - 4\cos^2\alpha}$, then they will not in fact be able to collide.
[Your solution must include a clear sketch.]

[12 marks]

2. (a) A home ornament consists of four metal butterflies of equal mass m supported from a light inextensible string of length L. The points of support are evenly spaced a distance ℓ apart as shown in the sketch below. The string forms an angle θ_1 with the ceiling at each end point.The centre section of string is horizontal.

   i. Find the tension in each section of the string in terms of θ_1, m, and g.



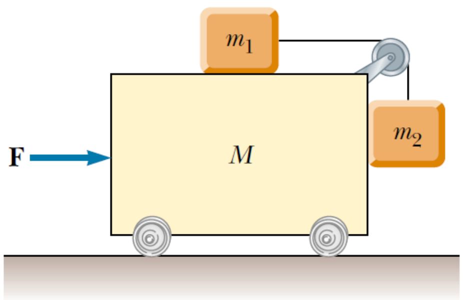
ii. Find the angle θ_2, in terms of θ_1, that the sections of string between the outside butterflies and the inside butterflies form with the horizontal.

iii. Show that the distance D between the end points of the string is

$$D = \frac{L}{5}\left\{2\cos\theta_1 + 2\cos\left[\tan^{-1}\left(\frac{1}{2}\tan\theta_1\right)\right] + 1\right\} .$$

[15 marks]

(b) What horizontal force must be applied to the cart shown below so that the blocks remain stationary relative to the cart? Assume all surfaces, wheels and pulleys are frictionless.
[5 marks]



3. A particle P of mass m, lies on a smooth table and is attached by a light inextensible string of length 2ℓ passing through a small hole O in the table to a particle Q, of mass m, hanging vertically. Initially OP is of length ℓ and P is projected at right angles to OP in the plane of the table with speed $(8g\ell/3)^{1/2}$. Let r be the distance of P from the hole at time t.

(a) Show that

$$\dot{r}^2 = \frac{7g\ell}{3} - \frac{4g\ell^3}{3r^2} - gr .$$

(b) Will Q move up? Will it reach O? Justify your answer.

(c) Show that the (variable) tension in the string is given by

$$T = \frac{mg}{2}\left[1 + \frac{8}{3}\left(\frac{\ell}{r}\right)^3\right].$$

[Hint: Use the equations of motion for a particle under a central force, together with the energy equation.]

[18 marks]

4. A spherical hailstone falls freely from rest under gravity. It initially has a radius a. As it falls its volume increases through condensation at a rate equal to λ times its surface area, where λ is a constant.

   (a) Show that after a time t its radius is equal to $a + \lambda t$.

   (b) Show that the velocity v at time t satisfies the differential equation

   $$\frac{dv}{dt} = g - \frac{3\lambda v}{r}.$$

   (c) Hence show that

   $$4\lambda v = g\left[a + \lambda t - a^4(a + \lambda t)^{-3}\right].$$

[10 marks]

5. (a) Consider the two-dimensional dynamical system

   $$\dot{x} = x - y - x^3 - y^2 x, \tag{1a}$$

   $$\dot{y} = x + y - x^2 y - y^3, \tag{1b}$$

   where $x = x(t)$ and $y = y(t)$ are functions of time t, while the dot indicates differentiation with respect to this variable.

   i. Express (1) in polar coordinates (r, θ), where $r = r(t)$ and $\theta = \theta(t)$.

   ii. Show that (1) has a cycle of period 2π, and then deduce that all trajectories of (1) are attracted by this cycle, provided they don't start at the origin $(x, y) = (0, 0)$.

[14 marks]

   (b) In standard notation, the motion of a particle is described by the initial-value problem

   $$\frac{d\mathbf{r}}{dt} - \frac{2}{t}\mathbf{r} = \mathbf{q}, \quad \mathbf{r}(1) = 2\mathbf{i} - \mathbf{j} \quad (t > 0), \tag{2}$$

   where $\mathbf{q}$ is a constant vector.

   i. Without solving the above ODE show that the acceleration $\mathbf{a}$ satisfies

   $$\mathbf{a} = 2t^{-2}(\mathbf{r} + \mathbf{q}t).$$

   ii. If $\mathbf{q} = \mathbf{i} + 4\mathbf{j}$ solve the initial-value problem (2) and find the position of the particle at $t = 3$.

[11 marks]