



Mathematics 1X

Calculus Examples

1. SETS, NUMBERS, INEQUALITIES

1. For each of the following pairs of sets \mathcal{A} and \mathcal{B} , find $\mathcal{A} \cap \mathcal{B}$:

(a) $\mathcal{A} = (-\infty, -3) \cup [5, \infty)$, $\mathcal{B} = (-\infty, -1) \cup [2, 9)$;

(b) $\mathcal{A} = (-\infty, -5) \cup [-2, 1) \cup [3, 7]$, $\mathcal{B} = (-\infty, -4] \cup (-2, 2) \cup [4, \infty)$;

(c) $\mathcal{A} = \{(-\infty, -3) \cup (4, \infty)\} \cap \{(-\infty, -2] \cup (-1, 8]\}$,
 $\mathcal{B} = (-\infty, -2) \cup (-1, 6] \cup (9, \infty)$;

(d) $\mathcal{A} = (-\infty, -3] \cup (1, 6] \cup (9, \infty)$, $\mathcal{B} = [-6, -5) \cup (-2, 2) \cup (4, 10]$.

Questions 2 through 8 are meant to provide familiarity with the set-builder notation.
Most of them require very little time to solve.

2. Find the sets \mathcal{A} and \mathcal{B} such that the following conditions are true:

- $\mathcal{A} \cup \mathcal{B} = \{x \in \mathbb{N} : x \leq 11\}$,
- $\mathcal{A} \cap \mathcal{B} = \{x \in \mathbb{N} : 7 \leq x \leq 10\}$,
- $\mathcal{B} \setminus \mathcal{A} = \{4, 5, 6, 11\}$.

3. The same question as above if the conditions the two sets must satisfy are changed to:

- $\mathcal{A} \cup \mathcal{B} = \{x \in \mathbb{N} : x \leq 9\}$,
- $\mathcal{B} \setminus \mathcal{A} = \{x \in \mathbb{N} : 4 \leq x \leq 8\}$,
- $\{3, 9\} \cap \mathcal{B} = \emptyset$,
- $\mathcal{A} \cap \mathcal{B} = \{1\}$.

4. Given that $\mathcal{A} = \{x \in \mathbb{N} : x \leq 5\}$, find all subsets $\mathcal{B} \subset \mathcal{A}$ such that $\mathcal{B} \cap \{1, 2, 3\} \neq \emptyset$.



5. Let $\mathcal{A} = \{1, 3, 5, 7\}$ and $\mathcal{B} = \{5, 7, 8, 9\}$. Find the set X which has 4 elements and satisfies the following conditions:

- $(\mathcal{A} \setminus \mathcal{X}) \cup (\mathcal{X} \setminus \mathcal{B}) = \{1, 3\}$,
- $\{3, 9\} \subset \mathcal{X}$.

6. Give an example of three sets \mathcal{A} , \mathcal{B} and \mathcal{C} such that

$$\mathcal{A} \subset \mathcal{B} \cup \mathcal{C}, \quad \mathcal{B} \subset \mathcal{A} \cup \mathcal{C}, \quad \mathcal{C} \subset \mathcal{A} \cup \mathcal{B}.$$

7. Find the sets \mathcal{A} , \mathcal{B} and \mathcal{C} such that the following conditions are simultaneously met:

- $\mathcal{A} \cap \mathcal{B} = \mathcal{B} \cap \mathcal{C} = \{2, 3\}$,
- $\mathcal{A} \setminus \mathcal{C} = \{1, 4, 6\}$,
- $\mathcal{B} \setminus \mathcal{C} = \{5, 7, 9\}$,
- $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} = \{x \in \mathbb{N} : x \leq 9\}$.

8. Solve the equation

$$\{a, b\} \cup \mathcal{X} = \{a, b, c\} \cap \mathcal{X},$$

where $a, b, c \in \mathbb{R}$ are given distinct numbers, and \mathcal{X} is a set which you must find.

9. Solve the following inequalities expressing your answer as union of intervals:

- (a) $\frac{1}{x-1} - \frac{1}{x+3} > 0$; (b) $\frac{1}{2x-3} \leq \frac{1}{2x+5}$; (c) $\frac{x+3}{3x^2+10x+3} < 0$;
- (d) $\frac{x+4}{x-2} < \frac{2}{x+1}$; (e) $\frac{1}{x+1} - \frac{1}{x} \leq \frac{1}{x-1} - \frac{1}{x-2}$;
- (f) $\frac{x^2-x+1}{x-1} + \frac{x^2-3x+1}{x-3} > 2x - \frac{1}{4x-8}$; (g) $\frac{1}{x+5} + \frac{1}{x-7} + \frac{1}{x-5} + \frac{1}{x+7} > 0$;
- (h) $x^3 - \frac{1}{x^3} \geq 4 \left(x - \frac{1}{x}\right)$; (i) $\frac{1}{x^2+x} \leq \frac{1}{2x^2+2x+3}$; (j) $x^2 + \frac{4x^2}{(x-2)^2} \leq 5$.

10. Find the solution of the following inequalities:

- (a) $\frac{2}{|x+2|} \leq 1$; (b) $\frac{2}{|x-2|} > \left| \frac{-3}{2x-1} \right|$; (c) $\frac{x^2-4x+4}{x^2-6x+9} + \frac{|x-2|}{|x-3|} < 12$;
- (d) $|x-1| + 2x \leq |x|$; (e) $|x^2-4| \leq |x^2-16|$;



(f) $(x-1)|x| + (x+1)|x-1| \leq 1$; (g) $|x^2 - x - 1| \leq x$;

(h) $\sqrt{\left(\frac{x+1}{3-2x}\right)^2} > 1$; (i) $20\sqrt{(x^2-x)^2} < 1$.

11. Write the solution of the following inequalities as union of intervals:

(a) $\frac{6x^3 - 12x^2 - 30x}{(x-4)(x+1)} \geq 3x$; (b) $\frac{2x^2 - 5x}{(x+2)(x-3)} \leq \frac{x}{2}$;

(c) $\frac{3x^3 + 4x^2 + 22}{(x+5)(x+1)} \geq 2x$; (d) $\frac{3x^2 - 9x - 2}{(x+1)(x-3)} \geq 2$;

[Hint: These questions are taken from the Degree Exams & Resits 2008/2010. You can find detailed solutions on Moodle.]

12. Write the following sets as union of intervals:

$$\mathcal{A}_1 = \{x \in \mathbb{R} : (x+1)(x-3)(x-4)(x^2+8) \geq 0\}; \quad \mathcal{A}_2 = \left\{x \in \mathbb{R} : \frac{x+3}{x-1} \leq x\right\};$$

$$\mathcal{A}_3 = \left\{x \in \mathbb{R} : \frac{1}{x+2} \geq \frac{x-3}{x^2+1}\right\}; \quad \mathcal{A}_4 = \left\{x \in \mathbb{R} : \left|\frac{x}{x-3}\right| \leq 2\right\}$$

$$\mathcal{A}_5 = \left\{x \in \mathbb{R} : \frac{x^2 - 3x + 10}{(x+1)(x-3)} \leq 4\right\}$$

13. Draw rough sketches (using shading) on separate little diagrams in \mathbb{R}^2 of the following sets [Do not find points of intersection]:

$$\mathcal{A} = \{(x, y) \in \mathbb{R}^2 : -2x + y \leq 8\};$$

$$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 : x + y < 5\};$$

$$\mathcal{C}_1 = \mathcal{A} \cap \mathcal{B}; \quad \mathcal{C}_2 = \mathcal{A} \cup \mathcal{B};$$

$$\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : |-2x + y| \leq 8\};$$

$$\mathcal{E} = \{(x, y) \in \mathbb{R}^2 : |x + y| \leq 5\};$$

$$\mathcal{F}_1 = \mathcal{D} \cap \mathcal{E}; \quad \mathcal{F}_2 = \mathcal{D} \cup \mathcal{E}.$$

14. In the xOy plane, indicate the points satisfying each of the following inequalities:

(a) $|x - y| \leq 1$; (b) $|x + y| > 2$; (c) $|x| - |y| \geq 1$;



(d) $|x| + |y| \leq 3$; (e) $|x - 1| + |y + 1| \geq 2$; (f) $|x - y| + |x + y| \leq 2$.

[Hint: A similar question to (d) together with its solution appears in the First Class Test 2009 -- see Moodle. Questions (e) and (f) are somewhat similar.]

2. PROBLEMS INVOLVING QUADRATICS

1. Complete the square in each of the following quadratics:

(a) $x^2 - 6x - 40$; (b) $3x^2 + 12x - 7$; (c) $10 - 4x - 2x^2$;

(d) $x^2 + x + 1$; (e) $x^2 - xy + y^2$; (f) $2 + x - 6x^2$.

2. By considering discriminants, say whether the following curves cut the x -axis [answer 'yes' or 'no' to each]:

(a) $y = x^2 - 6x + 10$; (b) $y = 4x^2 + x - 7$; (c) $y = 1 - 4x - 2x^2$;

(d) $y = x^2 - x + 1$; (e) $y = 3 - 5x - 12x^2$.

3. For each of the following, use completion of the square to identify the point at which the curve reaches its maximum or minimum value and sketch the curve roughly, indicating the points (if any) where the curve cuts the coordinate axes

(a) $y = x^2 - 6x - 40$; (b) $y = 3 - 2x - x^2$; (c) $y = 20x - 4x^2$;

(d) $y = x^2 + x + 1$.

State the images of the functions associated with these curves.

4. For what values of $\alpha \in \mathbb{R}$ is the inequality

$$\frac{\alpha x^2 + 3x + 4}{x^2 + 2x + 2} < 5$$

satisfied for all values $x \in \mathbb{R}$?

5. Find all values of the parameter $\beta \in \mathbb{R}$ for which the inequality

$$\left| \frac{x^2 + \beta x + 1}{x^2 + x + 1} \right| < 3$$

is satisfied for all values of $x \in \mathbb{R}$?



3. MAPPINGS (DOMAIN, CODOMAIN, IMAGE/RANGE)

1. Find the maximal domain of definition for the following two functions:

$$f(x) = \sqrt{\frac{3-3x}{15-2x-x^2}} - 1; \quad g(x) = \frac{\sqrt{3-x}}{\sqrt{x - \frac{x^2+2x-2}{3+2x-x^2}}}.$$

2. Recall that if $f : A \rightarrow B$ ($A, B \subset \mathbb{R}$) is a given function and $A' \subset A$ then, by definition, the **image of A' under f** is the set

$$f(A') := \{f(x) : x \in A'\}.$$

If $A' = A$ then the set $f(A)$ is called the **image of the function f** and is usually denoted by $\text{Im}(f)$.

Consider the following simple situation: $A = B = \mathbb{R}$ and $f(x) = 2x + 1$. Find the image under f of the following sets: (a) $[-1, 1]$; (b) $[-2, 0]$; (c) $(-2, 1]$; (d) $(1, 2)$.

3. State the maximal domain and image of each of the following functions:

$$(a) f_1(x) = \frac{1}{x^3}; \quad (b) f_2(x) = \sqrt{x^2 - 6x - 40}; \quad (c) f_3(x) = \frac{1}{x^2 - 4};$$

$$(d) f_4(x) = \frac{1}{x^2 + 1}.$$

4. Consider the following quadratic function, $f(x) = x^2 - 4x + 3$. Calculate the image under this function of the intervals $I_1 = [-4, 0]$ and $I_2 = [1, 6]$.
5. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^2 - 2x$. Calculate the following sets: (a) $f([0, 2])$; (b) $f([-2, 4])$; (c) $f([1, 6])$; (d) $f([-10, 0])$.
6. Determine $\text{Im}(f)$ for the following functions:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = x^2 + 5x - 6;$$

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = x^2 - x + 1;$$

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = \frac{x^2 - 2x - 3}{x^2 + x + 1};$$



$$f : \mathbb{R} \setminus \{1, 3\} \rightarrow \mathbb{R}, \quad f(x) = \frac{2x}{x^2 - 4x + 3};$$

$$f : \mathbb{R} \setminus \left\{ \frac{a+b}{2} \right\} \rightarrow \mathbb{R}, \quad f(x) = \frac{x^2 - ab}{2x - a - b} \quad (a, b \in \mathbb{R} \text{ given numbers}).$$

7. Find all linear functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $(f \circ f)(x) = x$ for all $x \in \mathbb{R}$.

8. Functions $f, g, h : \mathbb{R} \rightarrow \mathbb{R}$ are defined by

$$f(x) = x^2, \quad g(y) = \sin y, \quad h(z) = z^3.$$

Find simple formulas for the compositions $f \circ g \circ h$ and $h \circ g \circ f$. Do they agree for all values of x and should they?

9. Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 3x + 1$ and $g(x) = x + 3$. Solve the equation $(g \circ f \circ f)(x) = (f \circ g \circ g)(x)$.

10. Solve the equation $(f \circ f \circ f)(x) = x$, where

$$f(x) = \frac{ax + 1}{x + a}, \quad (a \in \mathbb{R}).$$

11. Consider $f, g : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = 2x^2 - 1$ and $g(x) = 4x^3 - 3x$. Check that $f \circ g = g \circ f$.

12. The function $f : \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$f(x) = \begin{cases} 4 + x, & \text{if } x \leq 0, \\ 4 - x, & \text{if } x > 0. \end{cases}$$

Find a simple expression for $f \circ f$.

[Hint: Because the function has two different branches, calculating $f \circ f$ requires us to solve the inequalities $f(x) > 0$ and $f(x) \leq 0$ before we substitute $f(x)$ for x in the expression of the function.]

13. Consider the functions

$$f : \mathbb{R} \rightarrow \mathbb{R} \quad f(x) = \begin{cases} 2x - 3, & \text{if } x \leq 0, \\ 7x, & \text{if } x > 0, \end{cases}$$

and

$$g : \mathbb{R} \rightarrow \mathbb{R} \quad g(x) = \begin{cases} x^2, & \text{if } x \leq -2, \\ 2x - 1, & \text{if } x > -2. \end{cases}$$



Find $f \circ g$ and $g \circ f$.

[Hint: Something similar as before. For instance, to find $g \circ f$ we must find the values of x for which $f(x) \leq -2$ and $f(x) > -2$, etc.]

14. Consider the function given by the expression

$$f(x) = \frac{x^2 + ax + 1}{x^2 - x + 1}.$$

Determine $a \in \mathbb{R}$ such that $\text{Im}(f) \subseteq [-3, 2]$. Are there any values of $a \in \mathbb{R}$ for which $\text{Im}(f) = [-1, 1]$? Justify your answer.

4. INJECTIVITY, SURJECTIVITY & BIJECTIVITY

1. Show that the function

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = ax^4 + bx + c, \quad (a \neq 0)$$

is not injective (a, b, c are given real numbers).

2. Prove that all linear functions $f : \mathbb{R} \rightarrow \mathbb{R}$ are bijective. For which such functions $f = f^{-1}$?

3. Is the function $g : \mathbb{R} \rightarrow \mathbb{R}$, $g(x) = x^3 + x + 2$ injective? Justify your answer.

4. Let $f, g : \mathbb{C} \rightarrow \mathbb{C}$ such that

$$g(x) = f(x) + f(\omega x) + f(\omega^2 x), \quad (\forall x \in \mathbb{C},$$

where $\omega \in \mathbb{C} \setminus \mathbb{R}$ with $\omega^3 = 1$. Show that g is not injective.

5. Let

$$f : (-1, 0) \rightarrow \left(\frac{5}{11}, \frac{1}{2}\right), \quad f(x) = \frac{2x^2 + 3}{5x^2 + 6}.$$

Show that the function is bijective.

6. Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^3 + x$.

(a) Establish the injectivity of f .

(b) *Assuming* that the function is surjective (this fact is non-trivial – don't waste time trying to prove it!), solve the equation $f(x) = f^{-1}(x)$, where the right-hand side of this last equality denotes the inverse of f .



7. Show that the following mappings are bijective and give the definition of their inverses:

(a) $f : [0, \infty) \rightarrow (0, 1]$,

$$f(x) = \frac{1}{1 + 4x^2}, \quad (\forall x \in [0, \infty));$$

(b) $g : [32, 212] \rightarrow [0, 100]$,

$$g(x) = \frac{5}{9}(x - 32), \quad (\forall x \in [32, 212]);$$

(c) $h : [3, \infty) \rightarrow [-49, \infty)$,

$$h(x) = x^2 - 6x - 40, \quad (\forall x \in [3, \infty)).$$

8. Consider the function $f : \mathbb{N} \rightarrow \mathbb{N}$, defined by

$$f(n) = \begin{cases} n + 1, & \text{if } n \text{ is even} \\ n - 1, & \text{if } n \text{ is odd.} \end{cases}$$

Show that this function is bijective and find its inverse.

9. Consider the function

$$f : \mathbb{R} \setminus \{1\} \rightarrow \mathbb{R} \setminus \{1\}, \quad f(x) = \frac{x + 1}{x - 1}, \quad (\forall x \in \mathbb{R} \setminus \{1\}).$$

Show that this function is bijective and $f = f^{-1}$, where f^{-1} denotes the inverse of f .

10. Let $A, B, C \subset \mathbb{R}$ be given sets, and consider the functions $f : A \rightarrow B$ and $g : B \rightarrow C$.

(a) If f and g are injective then $g \circ f$ is injective.

(b) If f and g are surjective then $g \circ f$ is surjective.

11. Show that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2, & \text{if } x \geq 2, \\ 3x - 2, & \text{if } x < 2 \end{cases}$$

is bijective and find f^{-1} .

12. Consider the functions $f : A \rightarrow B$ and $g : B \rightarrow C$, where $A, B, C \subset \mathbb{R}$.



- (a) If both f and g are bijective then $g \circ f$ has the same property.
- (b) Show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
- (c) Let $F : [1, \infty) \rightarrow [1, \infty)$ be defined by $F(x) = x^6 - 3x^5 + 6x^4 - 7x^3 + 6x^2 - 3x + 1$. Check that $F(x) = (x^2 - x + 1)^3$ and then use the previous two results to establish that F is bijective. Finally, find its inverse.

5. TRIGONOMETRIC FUNCTIONS & EQUATIONS

1. The questions that follow require you to establish an *identity*. This is usually proved by evaluating the RHS and LHS separately and then comparing the two results.

- (a) $(1 + \tan^2 A)(1 - \sin^2 A) = 1$;
- (b) $\tan A + \cot A = \sec A \operatorname{cosec} A$;
- (c) $\frac{1}{1 - \sin A} + \frac{1}{1 + \sin A} = 2 \sec^2 A$;
- (d) $(\tan A + \sec A)^2 = \frac{1 + \sin A}{1 - \sin A}$;
- (e) $\frac{\tan A + \cot B}{\cot A + \tan B} = \frac{\tan A}{\tan B}$;
- (f) $(1 + \tan A)^2 + (1 + \cot A)^2 = (\sec A + \operatorname{cosec} A)^2$;
- (g) $\sin\left(A + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}(\sin A + \cos A)$;
- (h) $\tan\left(A + \frac{\pi}{4}\right) = \frac{1 + \tan A}{1 - \tan A}$;
- (i) $\sin 3A \cos A - \cos 3A \sin A = \sin 2A$;
- (j) $\cos 4A \cos A - \sin 4A \sin A = \cos 3A \cos 2A - \sin 3A \sin 2A$;
- (k) $\frac{\sin 2A}{1 + \cos 2A} = \tan A$;
- (l) $\frac{\cos 2A}{1 + \sin 2A} = \tan\left(\frac{\pi}{4} - A\right)$;
- (m) $\frac{1 - \cos 2A + \sin 2A}{1 + \cos 2A + \sin 2A} = \tan A$;
- (n) $(2 \cos A + 1)(2 \cos A - 1) = 2 \cos 2A + 1$;



- (o) $\frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A} = 2$;
- (p) $\cos^2 A + \cos^2 \left(\frac{\pi}{3} + A\right) + \cos^2 \left(\frac{\pi}{3} - A\right) = \frac{3}{2}$;
- (q) $\cos 3A \sin 2A - \cos 4A \sin A = \cos 2A \sin A$;
- (r) $\cos 5A \cos 2A - \cos 4A \cos 3A = -\sin 2A \sin A$;
- (s) $\sin^2 \left(\frac{\pi}{8} + \frac{A}{2}\right) - \sin^2 \left(\frac{\pi}{8} - \frac{A}{2}\right) = \frac{1}{\sqrt{2}} \sin A$.

2. Given that

$$\cos A = \frac{4}{5} \quad \text{and} \quad \cos B = \frac{3}{5},$$

find $\cos(A - B)$.

3. It is known that

$$\sin A = \frac{3}{5} \quad \text{and} \quad \cos B = \frac{12}{13}.$$

Find $\cos(A + B)$.

4. (a) Show that

$$\cos \left(\frac{A}{3}\right) + \cos \left(\frac{A + 2\pi}{3}\right) + \cos \left(\frac{A + 4\pi}{3}\right) = 0.$$

(b) Use the formula for the cosine of a sum of two angles and other appropriate known results to establish that

$$4 \cos^3 A = \cos 3A + 3 \cos A.$$

(c) Use the previous results to simplify as much as possible the expression

$$\cos^3 \left(\frac{A}{3}\right) + \cos^3 \left(\frac{A + 2\pi}{3}\right) + \cos^3 \left(\frac{A + 4\pi}{3}\right).$$

5. Show that the expression

$$E_1 = \cos^2 6x + \sin^2 5x + \sin x \sin 11x$$

does not depend on x .

The same question for the expression

$$E_2 = \frac{\cos x}{1 + \sin x} + \frac{\sin x}{1 + \cos x} + \frac{(1 - \sin x) \cdot (1 - \cos x)}{\sin x \cdot \cos x}.$$



6. Given that $3 \sin B = \sin(2A + B)$ find $\tan(A + B) - 2 \tan A$.

7. It is known that

$$\cos A + \cos B = \alpha \quad \text{and} \quad \sin A + \sin B = \beta \quad (\alpha \neq 0, \alpha^2 + \beta^2 \neq 0).$$

Find $\cos(A + B)$ in terms of α and β .

8. Show that

$$(a) \frac{1}{\sin 10^\circ} - \frac{\sqrt{3}}{\cos 10^\circ} = 4; \quad (b) \frac{\cot 20^\circ - \cot 30^\circ}{\cot 30^\circ - \cot 40^\circ} = 2 \cos 20^\circ.$$

9. Show that

$$\cos^2 \frac{\pi}{16} + \cos^2 \frac{3\pi}{16} + \cos^2 \frac{5\pi}{16} + \cos^2 \frac{7\pi}{16} = 16 \cos \frac{\pi}{8} \cos \frac{3\pi}{8} \cos \frac{5\pi}{8} \cos \frac{7\pi}{8}.$$

[Hint: Use the double-angle formulae together with the sum-product formulae to transform the LHS.]

10. Show that

$$(a) \tan \frac{\pi}{24} = (\sqrt{2} - 1)(\sqrt{3} - \sqrt{2}); \quad (b) \cos \frac{2\pi}{5} = \frac{1}{4}(\sqrt{5} - 1).$$

11. If A, B, C are acute angles such that

$$\tan A = \frac{1}{2}, \quad \tan B = \frac{1}{5}, \quad \tan C = \frac{1}{8},$$

then show that $A + B + C = \pi/4$.

12. If A, B, C are the angles of a triangle, show that

$$(a) \sin^2 A + \sin^2 B + \sin^2 C = 2(1 + \cos A \cos B \cos C);$$

$$(b) \sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

[Hint: You'll need to remember what is the sum of the angles of a triangle. For (a) use the double-angle formulae to transform the LHS and then use the sum-product formulae. For (b) use directly the sum-product formulae.]

13. Find

$$(a) \sin \left(\cos^{-1} \left(-\frac{3}{5} \right) \right); \quad (b) \cos \left(2 \sin^{-1} \left(\frac{3}{4} \right) \right); \quad (c) \tan \left(\sin^{-1} \left(\frac{2}{3} \right) \right);$$

$$(d) \cos (\cot^{-1} (-2)); \quad (e) \cos (2 \tan^{-1} (2)) - \sin (4 \tan^{-1} (3)).$$



14. Prove that the function

$$f(x) = 2 \tan^{-1} x + \sin^{-1} \frac{2x}{1+x^2}$$

assumes the same values for all $x \geq 1$. Find that value.

15. For what values of $\alpha \in \mathbb{R}$ does the equation

$$\alpha \sin \left(x + \frac{\pi}{4} \right) = \sin 2x + 9$$

possess a solution?

16. Solve the following trigonometric equations:

(a) $\sin x = -\frac{1}{2}$; (b) $\sin \left(2x + \frac{\pi}{4} \right) = 1$; (c) $2 \sin x \cos x - 3 \sin 2x = 0$;

(d) $\sin \frac{x}{2} \cos \frac{\pi}{3} - \cos \frac{x}{2} \sin \frac{\pi}{3} = \frac{1}{4}$; (e) $\cos \left(3x - \frac{\pi}{6} \right) = -1$; (f) $\sin^4 \frac{x}{2} - \cos^4 \frac{x}{2} = \frac{1}{2}$;

(g) $\cos \frac{\pi}{6} \cos x - \sin \frac{\pi}{6} \sin x = \frac{\pi}{4}$; (h) $\cot \left(2x + \frac{\pi}{3} \right) = 2$; (i) $\cot \left(\frac{x}{2} - 3 \right) = -1$;

(j) $2 \sin x \cos x + \sqrt{3} - 2 \cos x - \sqrt{3} \sin x = 0$; (k) $\tan^2 x - 4 \tan x + 3 = 0$;

(l) $2 \tan x - 2 \cot x = 3$; (m) $\cos 2x - 3 \sin x + 2 = 0$.

17. Solve the following trigonometric equations:

(a) $\sin x = \cos 2x$; (b) $\sin 6x + \sin 4x = 0$; (c) $12 \cos x - 5 \sin x = -13$;

(d) $\sin 3x + \cos 2x = 1$; (e) $\sin x + \cos x = 1 - \sin 2x$;

(f) $\sin^4 x + \cos^4 x = \frac{7}{2} \sin x \cos x$; (g) $\tan x - \sin x = 1 - \tan x \sin x$.

18. Write each of the following as $R \cos(A + \varphi)$, where $R > 0$:

(a) $\cos A - \sin A$; (b) $-\sqrt{3} \cos A + \sin A$.

Use (b) to find the general solution of the equation

$$-\sqrt{3} \cos 2x + \sin 2x = \sqrt{3}.$$

19. Find the general solution of the equation

$$\sin 5x - \sin 3x = \sqrt{2} \sin x.$$



20. (a) Show that

$$\sin^2 A - \sin^2 B = \sin(A + B) \sin(A - B).$$

(b) Solve the equations:

(i) $\sin^2 x + \sin^2 2x = \sin^2 3x + \sin^2 4x$;

(ii) $\sin^2 x + \sin^2 2x = \sin^2 3x$.

21. (a) Using the double-angle formulae show that

$$\sin^4 A = \frac{1}{8} (3 - 4 \cos 2A + \cos 4A).$$

(b) Solve the equation:

$$\sin^4 x + \sin^4 \left(x + \frac{\pi}{4} \right) = \frac{1}{4}.$$

22. Solve the following trigonometric equations:

(a) $\tan x + \tan 2x + \tan 3x + \tan 4x = 0$;

(b) $(\sec x + \operatorname{cosec} x)\sqrt{2} = \sec^2 x + \operatorname{cosec}^2 x$;

(c) $\sec^4 x - \tan^4 x = 7$;

(d) $\sin^3 x + \cos^3 x = 1$.

6. LIMITS OF FUNCTIONS

1. Find the following limits

(a) $\lim_{x \rightarrow 1} \frac{x^2 - 1}{2x^2 - x - 1}$; (b) $\lim_{x \rightarrow 1} \frac{x^3 - 1}{x^3 - x^2 + x - 1}$; (c) $\lim_{x \rightarrow 1} \frac{x^2 - \sqrt{x}}{\sqrt{x} - 1}$;

(d) $\lim_{x \rightarrow \pi/6} \frac{2 \sin^2 x + \sin x - 1}{2 \sin^2 x - 3 \sin x + 1}$; (e) $\lim_{x \rightarrow 1} \left[\left(\frac{4}{x^2 - x^{-1}} - \frac{1 - 3x + x^2}{1 - x^3} \right)^{-1} + 3 \frac{x^4 - 1}{x^3 - x^{-1}} \right]$.

[Hint: most of these limits are of the form $\lim_{x \rightarrow x_0} (\dots)/(\dots)$ in which both the numerator and the denominator of the fractions are zero in the limit. Try to factor both the top and the bottom so that you can remove the common term $(x - x_0)$ before taking the limit.]



2. Make use of the *fundamental result*

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

to evaluate the following limits:

$$\begin{aligned} \text{(a)} \quad & \lim_{x \rightarrow 0} \frac{\sin 2x}{\sin 4x}; & \text{(b)} \quad & \lim_{x \rightarrow 0} \frac{1 - \cos 6x}{x^2}; & \text{(c)} \quad & \lim_{x \rightarrow 0} \frac{\tan 3x}{x}; \\ \text{(d)} \quad & \lim_{x \rightarrow 0} \frac{1 - \cos 6x}{x}; & \text{(e)} \quad & \lim_{x \rightarrow 0} \frac{\cos x - \cos 3x}{x^2}; \\ \text{(f)} \quad & \lim_{x \rightarrow 1} \frac{x^3 + x^2 - 2}{\sin(x-1)}; & \text{(g)} \quad & \lim_{x \rightarrow 0} \frac{\sin(a+2x) - 2\sin(a+x) + \sin a}{x^2}. \end{aligned}$$

3. By making use of the important result

$$P^2 - Q^2 = (P - Q)(P + Q), \quad (1)$$

evaluate the following limits:

$$\begin{aligned} \text{(a)} \quad & \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}; & \text{(b)} \quad & \lim_{x \rightarrow 0} \frac{\sqrt{2+5x} - \sqrt{2+x}}{x}; & \text{(c)} \quad & \lim_{x \rightarrow 4} \frac{\sqrt{x+5} - 3}{x-4}; \\ \text{(d)} \quad & \lim_{x \rightarrow 1} \frac{\sqrt{6x-5} - x}{x-1}; & \text{(e)} \quad & \lim_{x \rightarrow 1} \frac{\sqrt{5-x} - 2}{\sqrt{2-x} - 1}; & \text{(f)} \quad & \lim_{x \rightarrow 2} \frac{x - \sqrt{3x-2}}{x^2 - 4}. \end{aligned}$$

[Hint: You'll have to make suitable choices for P and Q in (1).]

4. Find the limits indicated below:

$$\begin{aligned} \text{(a)} \quad & \lim_{x \rightarrow 1} \frac{1-x^2}{\sin \pi x}; & \text{(b)} \quad & \lim_{x \rightarrow 0} \frac{x - \sin 2x}{x + \sin 3x}; & \text{(c)} \quad & \lim_{x \rightarrow 0} \left(\frac{\sin 2x}{x} \right)^{1+x}; \\ \text{(d)} \quad & \lim_{x \rightarrow 0} \frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{x}; & \text{(e)} \quad & \lim_{x \rightarrow 1} \frac{\sin \pi x}{\sin 3\pi x}; & \text{(f)} \quad & \lim_{x \rightarrow \pi} \frac{1 - \sin \frac{x}{2}}{\pi - x}; \\ \text{(g)} \quad & \lim_{x \rightarrow 1} (1-x) \tan \frac{\pi x}{2}; & \text{(h)} \quad & \lim_{x \rightarrow 1} \frac{\cos \frac{\pi x}{2}}{1 - \sqrt{x}}. \end{aligned}$$



7. DIFFERENTIATION

1. Using differentiation by first principles, find the derivative of each of the following functions:

(a) x^2 ; (b) x^3 ; (c) $\frac{1}{x}$;

(d) \sqrt{x} ; (e) $\frac{1}{x+1}$.

2. Using the Chain/Product/Quotient Rules, find the derivatives of the following functions:

(a) $3x^3 + x^2 - 4x + 8$; (b) $(3x + 1)^5$; (c) $(x^2 + 4)^3$; (d) $\sin 5x$;

(e) $\cos 2x$; (f) $x^2 \sin 3x$; (g) $x^2 \cos 2x$; (h) $\cos^2 x$;

(i) $\tan^2 x$; (j) $\frac{1}{(x+1)^4}$; (k) $\sqrt{x^2 + 1}$; (l) $\sqrt{\sin x}$;

(m) $\sec^2 x$; (n) $\frac{1}{\sqrt{x^2 + 1}}$;

3. Differentiate the following functions with respect to x , simplifying your answers as much as possible:

(a) $(x^2 + 1)^5$; (b) $x^3(3x + 1)^2$; (c) $\frac{x}{x^2 + 4}$; (d) $\frac{x}{\sqrt{x^2 + 4}}$;

(e) $\sin^4 2x$; (f) $x\sqrt{\cos 2x}$; (g) $\left(\frac{2x-1}{2x+1}\right)^3$; (h) $\sqrt{1 + \sin^2 x}$;

(i) $x^2 \tan x$; (j) $\sin^4 x \cos 4x$; (k) $\frac{\cos^3 x}{\cos 3x}$; (l) $x \cot x$;

(m) $(\sec x + \tan x)^3$; (n) $\frac{1}{\sec x + \tan x}$; (o) $\sin^{-1} 2x$; (p) $\cos^{-1} 2x$;

(q) $\operatorname{cosec}^2 x$; (r) $\tan^{-1}\left(\frac{x}{2}\right)$; (x) $\sqrt{1-x^2}$; (t) $x\sqrt{1-x^2}$; (u) $x\sqrt{1+2x} e^{2x}$.

4. Differentiate the following functions with respect to x , simplifying your answers as much as possible:

(a) $x \log x - x$; (b) $\log(x^2 + 1)$; (c) $\log\left(\frac{1+x}{1-x}\right)$; (d) $\log(\sec x + \tan x)$;



- (e) $e^{\sin 2x}$; (f) $xe^{-\frac{1}{2}x^2}$; (g) $\frac{e^x}{e^{2x} + 1}$; (h) $e^{(\log x)^2}$;
(i) 2^x ; (j) x^x ; (k) $x^{1/x}$; (l) $\tan^{-1}(e^x)$;
(m) $\tan^{-1}\left(\frac{1}{x}\right)$; (n) $\log(\log x)$; (n) $e^{-3x} \cos 2x$.

8. IMPLICIT DIFFERENTIATION

1. Find dy/dx in terms of x and y if

- (a) $xy - x + 2y = 1$; (b) $x^3 + y^3 = 1$; (c) $x^2 + xy = y^3$; (d) $x^3y + xy^5 = 2$;
(e) $x^2y^3 = 2x - y$; (f) $x^2 + 4(y - 1)^2 = 4$; (g) $\frac{x - y}{x + y} = \frac{x^2}{y} + 1$.

2. Find d^2y/dx^2 in terms of x and y if

- (a) $xy = x + y$; (b) $x^2 + 4y^2 = 4$;
(c) $x^3 - y^2 + y^3 = x$; (d) $x^3 - 3xy + y^3 = 1$.

3. If $Ax^2 + By^2 = C$, show that

$$\frac{d^2y}{dx^2} = -\left(\frac{AC}{B^2}\right) \frac{1}{y^3}.$$

4. Calculate d^2y/dx^2 at the point $(1, 1)$, if

$$x^2 + 5xy + y^2 - 2x + y - 6 = 0.$$

5. Calculate d^2y/dx^2 at the point $(0, 1)$, if

$$x^4 - xy + y^4 = 1.$$

6. A function $y = y(x)$ is defined implicitly by the equation

$$x^2 + 2xy + y^2 - 4x + 2y - 2 = 0.$$

Calculate d^3y/dx^3 at the point $(1, 1)$.



7. Given that $x^2 + y^2 = R^2$ find d^3y/dx^3 .

9. INTEGRATION:

9.1 standard types & variants

1. Find the following integrals

$$(a) \int \frac{1}{x^2 + 9} dx; \quad (b) \int \frac{1}{\sqrt{16 - x^2}} dx; \quad (c) \int \frac{1}{(x + 6)^3} dx;$$

$$(d) \int \frac{1}{x + 8} dx; \quad (e) \int \frac{x}{x + 4} dx; \quad (f) \int \frac{x}{(x + 5)^2} dx;$$

$$(g) \int \frac{x^2}{x^2 + 9} dx.$$

9.2 by parts

1. Use integration by parts to find the following integrals

$$(a) \int x e^{5x} dx; \quad (b) \int x \sin 2x dx; \quad (c) \int x \log x dx;$$

$$(d) \int x^4 \log x dx; \quad (e) \int (\log x)^2 dx; \quad (f) \int x^2 \sin x dx;$$

$$(g) \int (x^2 + x)e^{-x} dx; \quad (h) \int e^{-x} \sin 2x dx; \quad (i) \int e^{3x} \cos 2x dx.$$

2. Prove the following formulae

$$(a) \int x^2 e^x dx = e^x(x^2 - 2x + 2) + \text{Const.};$$

$$(b) \int x^2 e^{-x} dx = -e^{-x}(x^2 + 2x + 2) + \text{Const.}$$



9.3 by substitution or change of variable

1. By means of a change of variable or otherwise find the following:

(a) $\int \frac{1}{3x+1} dx$; (b) $\int (5x+4)^{-1/2} dx$; (c) $\int \cos(2x+3) dx$;

(d) $\int x\sqrt{x^2-4} dx$; (e) $\int \frac{x}{x^2+1} dx$; (f) $\int \frac{x}{(2-x)^{2/3}} dx$;

(g) $\int \frac{1}{4x^2+1} dx$; (h) $\int xe^{-x^2/2} dx$; (i) $\int \frac{1}{\sqrt{9x^2-1}} dx$;

(j) $\int x(4x^2+1)^{3/2} dx$; (k) $\int \sec^2 4x dx$; (l) $\int \frac{\sin x}{1+\cos^2 x} dx$;

(m) $\int x\sqrt{98x^2+4} dx$; (n) $\int \frac{1}{9x^2+4} dx$; (o) $\int \frac{1}{\sqrt{25-4x^2}} dx$;

(p) $\int x^2\sqrt{1+x^3} dx$; (q) $\int \sin^6 x \cos x dx$; (r) $\int x \sec^2(x^2) dx$.

2. The same question for the integrals included below: Use integration by parts to find the following integrals

(a) $\int x^3\sqrt{x^2+4} dx$; (b) $\int x \log(x^2+1) dx$; (c) $\int \frac{(\log x)^3}{x} dx$;

(d) $\int \frac{x^3}{x^4+16} dx$; (e) $\int \frac{x}{x^4+16} dx$; (f) $\int \frac{\sec^2 x}{\tan x} dx$

(g) $\int x^2\sqrt{x+2} dx$; (h) $\int x^3 e^{-x^2/2} dx$.

3. Use a suitable change of variable to determine the following:

(a) $\int \sin^2 \theta \cos \theta d\theta$; (b) $\int \cos^4 \theta \sin \theta d\theta$;

(c) $\int \sqrt{\sin \theta} \cos \theta d\theta$; (d) $\int \cos^4 \theta \sin^3 \theta d\theta$.



9.4 general problems

1. Find the integrals:

$$(a) \int \sin^2 \theta \, d\theta; \quad (b) \int \cos^2 \theta \, d\theta; \quad (c) \int \sin^2 \theta \cos^2 \theta \, d\theta;$$

$$(d) \int \cos^4 \theta \, d\theta; \quad (e) \int \cos^2 4\theta \, d\theta; \quad (f) \int \sin^6 \theta \, d\theta.$$

2. Find the following:

$$(a) \int \tan \theta \, d\theta; \quad (b) \int \tan^2 \theta \, d\theta; \quad (c) \int \cot 3\theta \, d\theta; \quad (d) \int \cot^2 2\theta \, d\theta.$$

3. Find the following:

$$(a) \int \sec \theta \, d\theta; \quad (b) \int \sec^2 \theta \, d\theta; \quad (c) \int \sec^3 \theta \, d\theta; \quad (d) \int \sec^4 \theta \, d\theta.$$

4. Find the following:

$$(a) \int \tan^{-1} x \, dx; \quad (b) \int x \tan^{-1} x \, dx; \quad (c) \int \sin^{-1} x \, dx;$$

5. Use trig substitutions or otherwise to find:

$$(a) \int \sqrt{a^2 - x^2} \, dx; \quad (b) \int (a^2 - x^2)^{3/2} \, dx; \quad (c) \int \frac{x}{(x+2)^2} \, dx;$$

$$(d) \int \sqrt{a^2 + x^2} \, dx; \quad (e) \int \frac{1}{\sqrt{x^2 - a^2}} \, dx.$$

(Here $a > 0$ is a given number).



6. Using partial fractions or otherwise find the following:

(a) $\int \frac{1}{x^2 - 4} dx$; (b) $\int \frac{x + 3}{(x - 3)(x + 1)} dx$; (c) $\int \frac{x}{(x + 2)^2} dx$;

(d) $\int \frac{x^2 + 2x + 4}{x(x + 1)^2} dx$; (e) $\int \frac{x - 14}{(x - 2)(x^2 + 8)} dx$; (f) $\int \frac{3x + 2}{(x + 1)(x^2 + 2x + 2)} dx$.

7. By completing the square, calculate the integrals:

(a) $\int \frac{x}{\sqrt{3 - 2x - x^2}} dx$; (b) $\int \frac{1}{\sqrt{2x - x^2}} dx$;

(c) $\int \frac{1}{\sqrt{x^2 - 4x + 13}} dx$; (d) $\int \frac{1}{x^2 - 4x + 8} dx$;

(e) $\int \frac{1}{x^2 - x + 1} dx$; (f) $\int \frac{1}{x^3 + 1} dx$.