

Inverse substitutions and other non-standard methods

The substitutions considered earlier were direct substitutions, in the sense that we simplified an integrand by replacing an expression appearing in it with a single variable. Here the reverse approach is considered: *the variable of integration is replaced with a function of a new variable*. Such substitutions are called **inverse substitutions**. They would appear on the surface to make the integral more complicated; that is, substituting $x = g(u)$ in the integral

$$\int f(x) dx$$

leads to the more “complicated” integral

$$\int f(g(u)) \cdot g'(u) du.$$

Sometimes such substitutions can actually simplify an integrand, transforming the integral into one that can be evaluated by inspection or to which other techniques can be readily applied.

Next, we are going to take a closer look at some standard inverse substitutions.

The inverse sine substitution: is usually useful if the integrand involves

$$\sqrt{a^2 - x^2}, \quad (a > 0).$$

The substitution itself is given by

$$x = a \sin \theta \quad \Leftrightarrow \quad \theta = \sin^{-1} \left(\frac{x}{a} \right). \quad (1)$$

Note that

$$\sqrt{a^2 - x^2} = \sqrt{a^2(1 - \sin^2 \theta)} = \sqrt{a^2 \cos^2 \theta} = a \cos \theta;$$

hence

$$\cos \theta = \frac{\sqrt{a^2 - x^2}}{a} \quad \text{and} \quad \tan \theta = \frac{x}{\sqrt{a^2 - x^2}}.$$

Here is how this type of substitution works:

$$\begin{aligned}
\int \frac{dx}{(5-x^2)^{3/2}} &= \int \frac{\sqrt{5} \cos \theta}{5^{3/2} \cos^3 \theta} d\theta \\
&= \frac{1}{5} \int \sec^2 \theta d\theta = \frac{1}{5} \tan \theta + \text{const.} \\
&= \frac{1}{5} \tan \left(\sin^{-1} \left(\frac{x}{\sqrt{5}} \right) \right) + \text{const.}
\end{aligned}$$

Let $x = \sqrt{5} \sin \theta$.
Then $dx = \sqrt{5} \cos \theta d\theta$.

The inverse tangent substitution: is usually useful if the integrand involves

$$\sqrt{a^2 + x^2} \quad \text{or} \quad x^2 + a^2, \quad (a > 0).$$

The substitution itself is given by

$$x = a \tan \theta \quad \Leftrightarrow \quad \theta = \tan^{-1} \left(\frac{x}{a} \right). \quad (2)$$

Note that

$$\sqrt{a^2 + x^2} = a\sqrt{1 + \tan^2 \theta} = a \sec \theta;$$

hence

$$\sin \theta = \frac{x}{\sqrt{x^2 + a^2}} \quad \text{and} \quad \cos \theta = \frac{a}{\sqrt{a^2 + x^2}}.$$

Here are a couple of representative examples:

$$\begin{aligned}
\int \frac{dx}{\sqrt{4+x^2}} &= \int \frac{2 \sec^2 \theta}{2 \sec \theta} d\theta \\
&= \int \sec \theta d\theta = \log |\sec \theta + \tan \theta| + \text{const.} \\
&= \log \left| \frac{\sqrt{4+x^2}}{2} + \frac{x}{2} \right| + \text{const.} \\
&= \log(\sqrt{4+x^2} + x) + \text{const.}
\end{aligned}$$

Let $x = 2 \tan \theta$.
Then $dx = 2 \sec^2 \theta d\theta$.

$$\begin{aligned}
\int \frac{dx}{(1+9x^2)^2} &= \frac{1}{3} \int \frac{\sec^2 \theta}{\sec^4 \theta} d\theta \\
&= \frac{1}{3} \int \cos^2 \theta d\theta = \frac{1}{6}(\theta + \sin \theta \cos \theta) + \text{const.} \\
&= \frac{1}{6} \tan^{-1}(3x) + \frac{1}{6} \cdot \frac{3x}{\sqrt{1+9x^2}} \cdot \frac{1}{\sqrt{1+9x^2}} + \text{const.} \\
&= \frac{1}{6} \tan^{-1}(3x) + \frac{x}{2(1+9x^2)} + \text{const.}
\end{aligned}$$

Let $3x = \tan \theta$.

Then $3dx = \sec^2 \theta d\theta$

and $1+9x^2 = 1 + \tan^2 \theta = \sec^2 \theta$.

Another useful substitution:

$$x = \tan \frac{\theta}{2} \quad \Leftrightarrow \quad \theta = 2 \tan^{-1} x. \quad (3)$$

Note that

$$\cos^2 \frac{\theta}{2} = \frac{1}{\sec^2 \frac{\theta}{2}} = \frac{1}{1 + \tan^2 \frac{\theta}{2}} = \frac{1}{1 + x^2},$$

and

$$d\theta = 2 \frac{d}{dx}(\tan^{-1} x) dx = \frac{2dx}{1+x^2}.$$

With this information in the double-angle formulae,

$$\begin{aligned}
\cos \theta &= 2 \cos^2 \frac{\theta}{2} - 1 = \frac{2}{1+x^2} - 1 = \frac{1-x^2}{1+x^2}, \\
\sin \theta &= 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = 2 \tan \frac{\theta}{2} \cos^2 \frac{\theta}{2} = \frac{2x}{1+x^2}.
\end{aligned}$$

To summarise,

$$\begin{aligned}
\cos \theta &= \frac{1-x^2}{1+x^2}, \\
\sin \theta &= \frac{2x}{1+x^2}, \\
d\theta &= \frac{2dx}{1+x^2}.
\end{aligned}$$

Here is a first example using this substitution:

$$\begin{aligned}\int \frac{d\theta}{2 + \cos \theta} &= \int \frac{1}{2 + \frac{1-x^2}{1+x^2}} \cdot \frac{2 dx}{1+x^2} \\ &= 2 \int \frac{dx}{3+x^2} = \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{x}{\sqrt{3}} \right) + \text{const.} \\ &= \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{1}{\sqrt{3}} \tan \frac{\theta}{2} \right) + \text{const.}\end{aligned}$$

$$\text{Let } x = \tan \frac{\theta}{2}.$$

$$\text{Then } dx = \frac{2 dx}{1+x^2}.$$

And here is another one:

$$\begin{aligned}\int \frac{d\theta}{1 + \cos \theta + \sin \theta} &= \int \frac{1}{1 + \frac{1-x^2}{1+x^2} + \frac{2x}{1+x^2}} \cdot \frac{2 dx}{1+x^2} \\ &= \int \frac{dx}{x+1} = \log |x+1| + \text{const.} \\ &= \log \left| 1 + \tan \frac{\theta}{2} \right| + \text{const.}\end{aligned}$$

$$\text{Let } x = \tan \frac{\theta}{2}.$$

$$\text{Then } dx = \frac{2 dx}{1+x^2}.$$