

Trigonometric integrals

This type of integrals requires a good knowledge of the trigonometric formulae mentioned in the earlier part of the course. In addition, the following *standard integrals* will have to be memorised:

$$\begin{aligned}\int \tan x \, dx &= \log |\sec x| + \text{const.} \\ &= -\log |\cos x| + \text{const.}\end{aligned}\tag{1a}$$

$$\begin{aligned}\int \cot x \, dx &= \log |\sin x| + \text{const.} \\ &= -\log |\operatorname{cosec} x| + \text{const.}\end{aligned}\tag{1b}$$

$$\int \sec x \, dx = \log |\sec x + \tan x| + \text{const.}\tag{1c}$$

$$\begin{aligned}\int \operatorname{cosec} x \, dx &= -\log |\operatorname{cosec} x + \cot x| + \text{const.} \\ &= \log |\operatorname{cosec} x - \cot x| + \text{const.}\end{aligned}\tag{1d}$$

Each one of the above integrals can be checked by differentiating the RHS's. It is possible to establish these results by using the integration methods studied so far. For example,

$$\begin{aligned}\int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx \\ &= -\int \frac{du}{u} = -\log |u| + \text{const.} \\ &= -\log |\cos x| + \text{const.}, \text{ etc}\end{aligned}$$

Let $u = \cos x$.

Then $du = -\sin x \, dx$.

The integral in (1c) is a lot harder to calculate directly:

$$\begin{aligned}
\int \sec x \, dx &= \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} \, dx \\
&= \int \frac{du}{u} = \log |u| + \text{const.} \\
&= \log |\sec x + \tan x| + \text{const.}, \text{ etc}
\end{aligned}$$

$$\begin{aligned}
\text{Let } u &= \sec x + \tan x . \\
\text{Then } du &= (\sec x)' \, dx + (\tan x)' \, dx \\
&= (\sec x \tan x) \, dx + (\sec^2 x) \, dx \\
&= \sec x(\sec x + \tan x) \, dx .
\end{aligned}$$

Integrals of the form:

$$\int \sin^m x \cdot \cos^n x \, dx .$$

There are two important cases:

- If either m or n is an **odd, positive integer**, then the integral can be done by substitution.

For the sake of the argument, let's say that $n = 2k + 1$ ($k \in \mathbb{N}$). Then

$$\begin{aligned}
\int \sin^m x \cdot \cos^n x \, dx &= \int \sin^m x \cos^{2k} x \cos x \, dx \\
&= \int \sin^m x (1 - \sin^2 x)^k \cos x \, dx .
\end{aligned}$$

If we now let $u = \sin x$, then $du = \cos x \, dx$ and the last integral reduces to

$$\int u^m (1 - u^2)^k \, du, \quad \text{etc.}$$

- If **both** m and n are **even**, then we can use the **double-angle formulae**:

$$\begin{aligned}
\cos^2 x &= \frac{1}{2}(1 + \cos 2x) , \\
\sin^2 x &= \frac{1}{2}(1 - \cos 2x) .
\end{aligned}$$

Ex. 1) $\int \sin^3 x \cos^8 x \, dx$

Ex. 2) $\int \cos^5(\alpha x) \, dx$

Ex. 3) $\int \sin^4 x \, dx$

Integrals of the form:

$$\int \sec^m x \cdot \tan^n x \, dx \quad \text{OR} \quad \int \operatorname{cosec}^m x \cdot \cot^n x \, dx$$

Integration by substitution will always work **unless**

m is odd **and** n is even.

In that case integration by parts can be applied.

The use of an appropriate substitution will invariably require knowledge of the following important formulae (which must be memorised):

$$\begin{aligned} \sec^2 x &= 1 + \tan^2 x, \\ \operatorname{cosec}^2 x &= 1 + \cot^2 x. \end{aligned}$$

Ex. 4) $\int \tan^2 x \, dx$

Ex. 5) $\int \sec^4 x \, dx$

Ex. 6) $\int \sec^3 x \tan^3 x \, dx$

Ex. 7) $\int \sec^6 x \tan^2 x \, dx$

Ex. 8) $\int \sec^5 x \tan x \, dx$

Further examples*:

$$\int \frac{\sin^2 x}{\cos^4 x} \, dx, \quad \int \frac{\sin^3 x}{\cos^4 x} \, dx.$$