Taylor's Theorem

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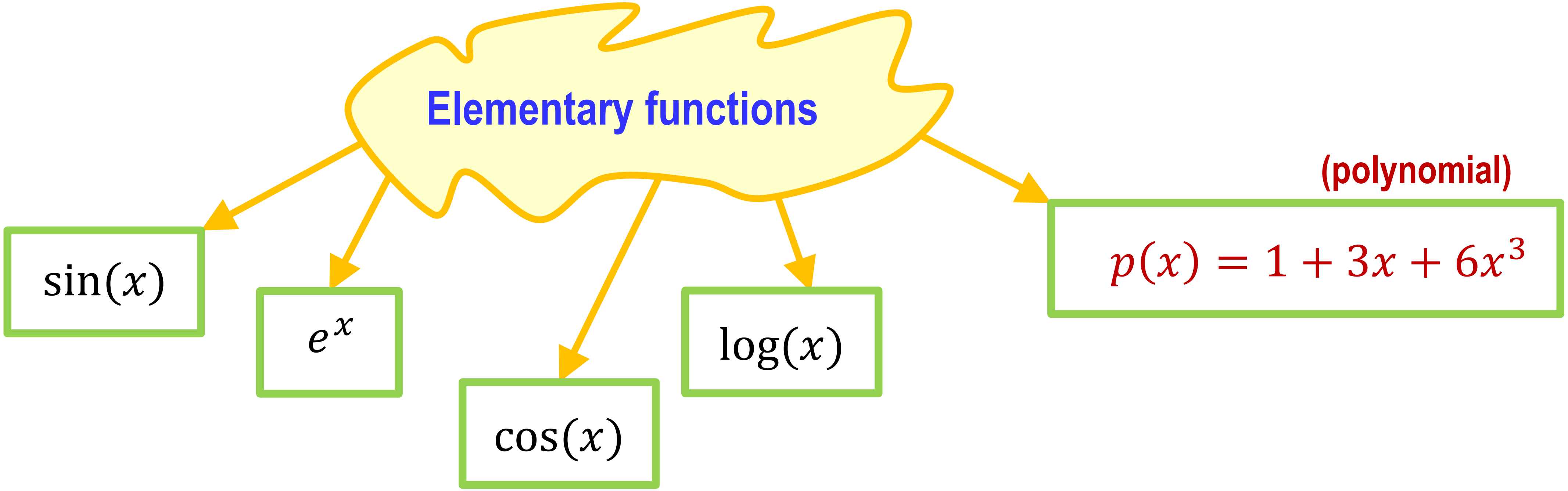
Outline

- Motivation/main idea
- **Taylor polynomials**
- Simple examples
- **Taylor's Theorem**

Learning Objective: introduce the key ideas & terminology related to Taylor's Theorem

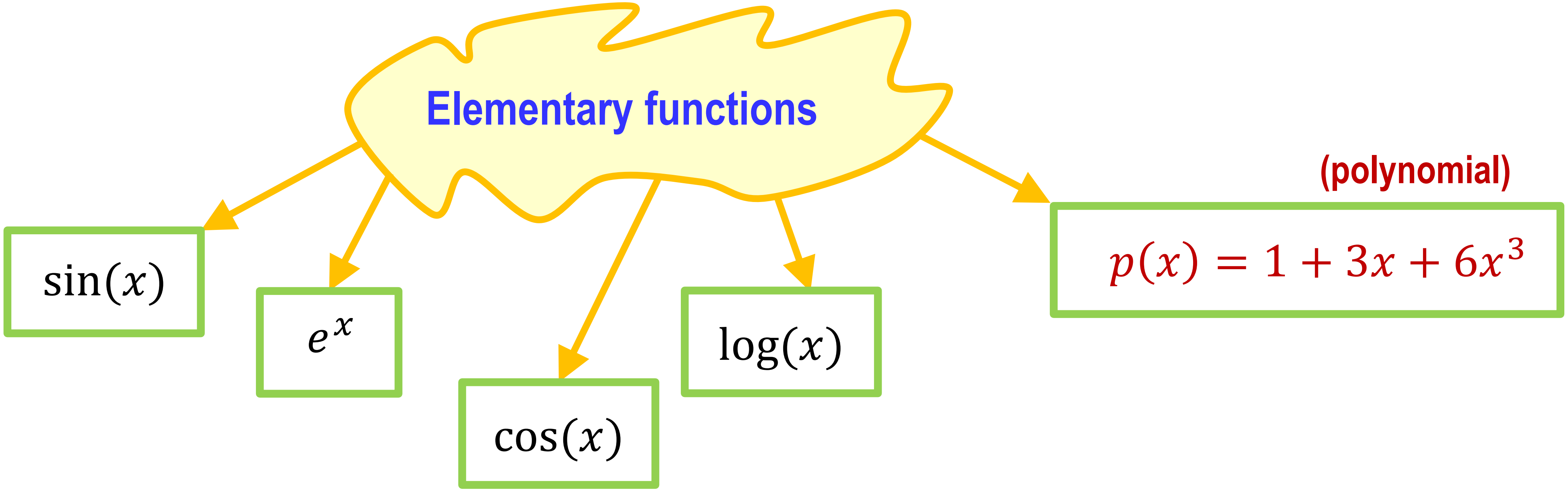
Motivation



Observation: In some sense, *some* of these functions are NOT elementary at all:

- Can you find **without a calculator** $e^{0.2}$ or $\sin(0.1213)$?
- On the other hand, it is very easy to evaluate $p(0.2)$ or $p(0.1213)$

Motivation



Observation: It is **very easy** to evaluate $p(0.2)$ or $p(0.1213)$ ….

$$p(0.2) = 1 + 3 \times (0.2) + 6 \times (0.2)^3 \approx 1.648$$

$$p(0.1213) = 1 + 3 \times (0.1213) + 6 \times (0.1213)^3 \approx 1.374608 \ldots.$$

} don't need a calculator

ASIDE

Given a **smooth** function, $f: I \subseteq \mathbb{R} \to \mathbb{R}$, can we find a **polynomial** $P(x)$ such that

$$f(x) \approx P(x) \qquad \text{for all } x \text{ close to } a \in I$$

What does it mean for $f(x)$ to be *"approximately equal to"* $P(x)$?

$$f(x) = P(x) + \underbrace{\varepsilon(x)} \quad \Leftrightarrow \quad \varepsilon(x) = f(x) - P(x)$$

- **remainder term** (also known as **error term**)
- typically, very small: $10^{-2}, 10^{-3}$, etc
- it depends on the value of x

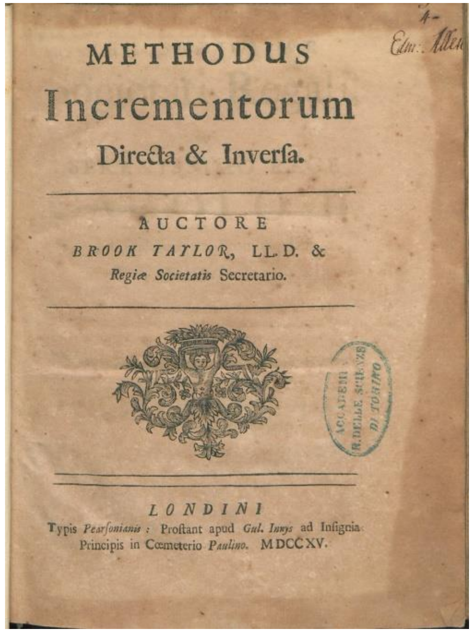
smooth = has as many derivatives as we like (e.g., infinitely differentiable)

The main idea

Given a smooth function, $f : I \subseteq \mathbb{R} \to \mathbb{R}$, can we find a **polynomial** $P(x)$ such that

$$f(x) \approx P(x) \qquad \text{for all } x \textbf{ close} \text{ to } a \in I \text{ ?}$$

Brook Taylor (1715)



$f(x)$

↓

$\arctan(x)$

$\operatorname{arcsec}(\sqrt{2}e^{x})$

$\log(\sec x)$

James Gregory (1671)

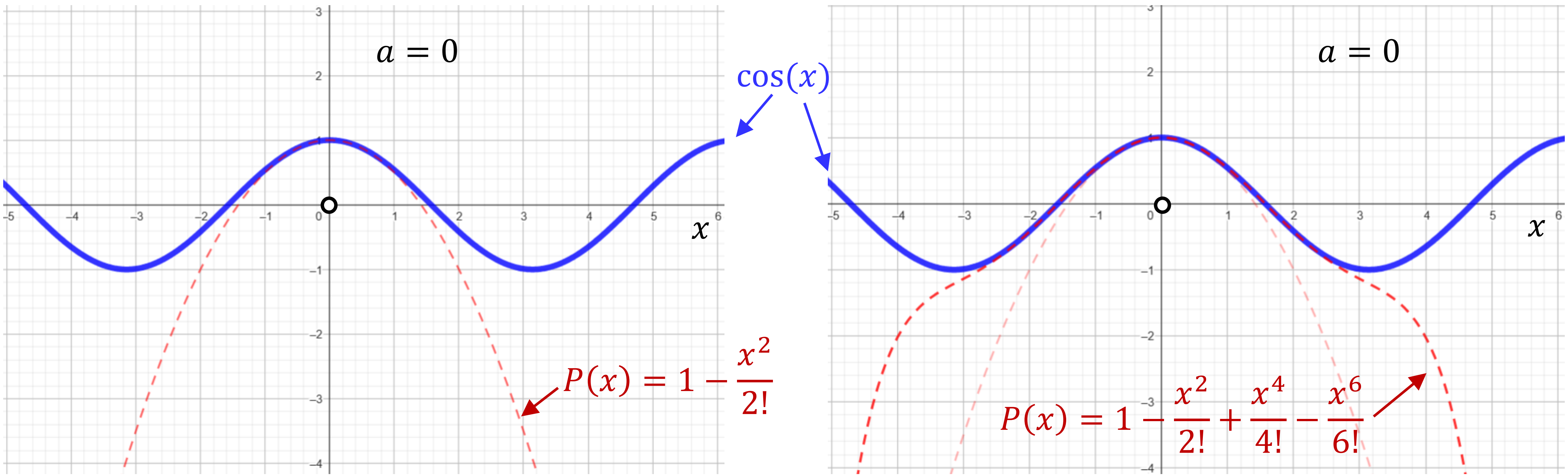
The main idea (concrete example)

Given the function $f(x) = \cos x$, can we find a **polynomial** $P(x)$ such that

$$\cos x \approx P(x) \qquad \text{for all } x \text{ close to } a = 0\ ?$$

EXAMPLE 1

EXAMPLE 2



Taylor polynomials (main definition)

Definition 1: Suppose that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a function *(not necessarily a polynomial)* such that

$$f^{(1)}(a), f^{(2)}(a), f^{(3)}(a), \ldots, f^{(n)}(a)$$

← The first n derivatives at $x = a$

all exist, where $a \in I \subseteq \mathbb{R}$ is a given value.
The **Taylor polynomial of degree n for f at a** is defined by

$$P_{n,a}(x) = f(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$b_0 + b_1(x-a) + b_2(x-a)^2 + \cdots + b_n(x-a)^n$$

a polynomial in which the coefficients depend on the derivatives of the given function $f(x)$

Taylor polynomials (1st example)

$$P_{n,a}(x) = f(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$f(x) = e^x \qquad (a = 0)$

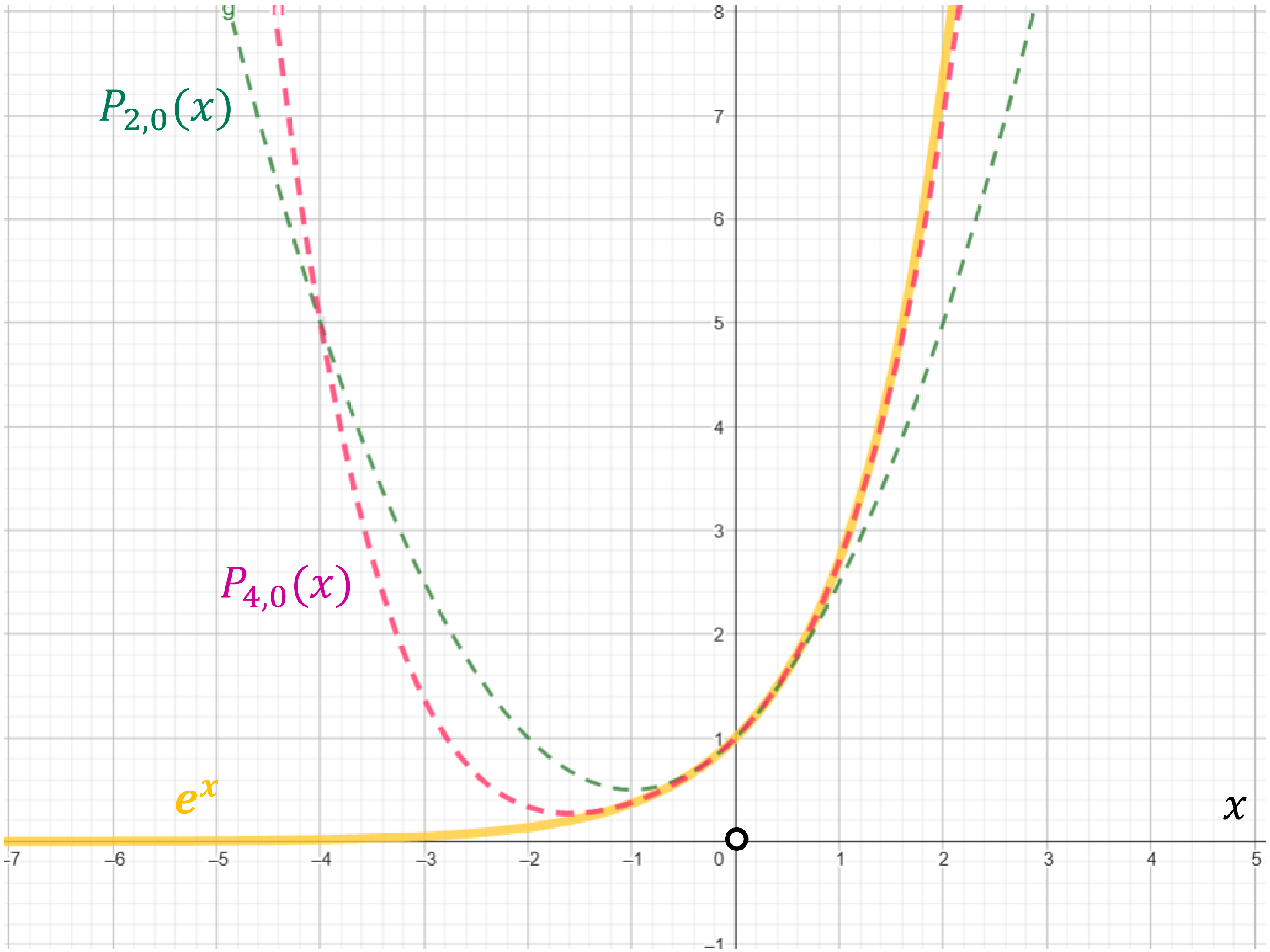
$$(e^x)' = e^x \quad \Rightarrow \quad \frac{f^{(k)}(0)}{k!} = \frac{e^0}{k!} = \frac{1}{k!}$$

Hence

$$P_{n,0}(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

$n = 2$: $\quad P_{2,0}(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!}$

$n = 4$: $\quad P_{4,0}(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$



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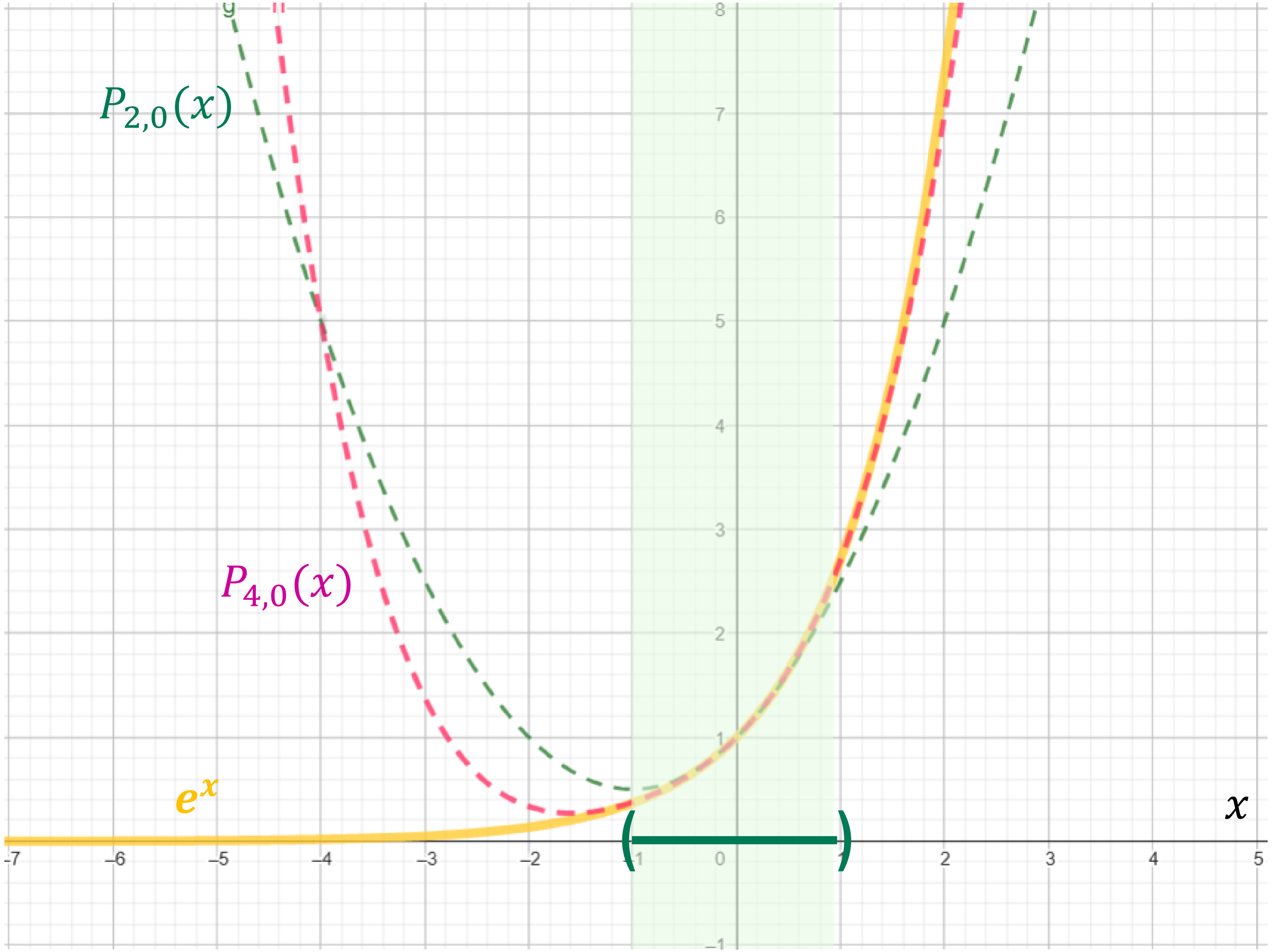

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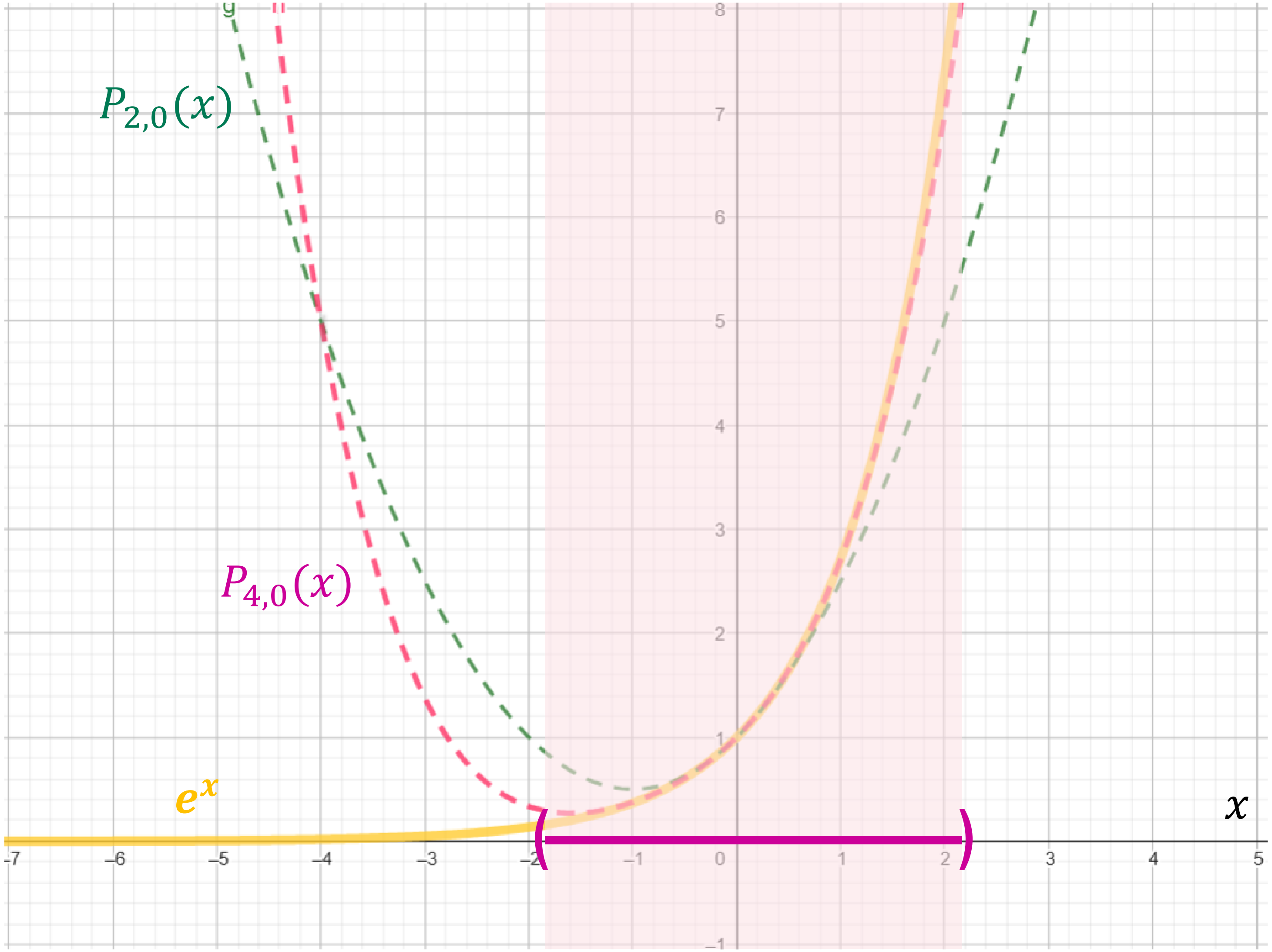


Taylor polynomials (2nd example)

$$P_{n,a}(x) = f(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$f(x) = \cos x \qquad (a = 0)$$

$\cos(0) = 1$

$cos^{(1)}(0) = -\sin(0) = 0$

$cos^{(2)}(0) = -\cos(0) = -1$

$cos^{(3)}(0) = \sin(0) = 0$

$cos^{(4)}(0) = \cos(0) = 1$

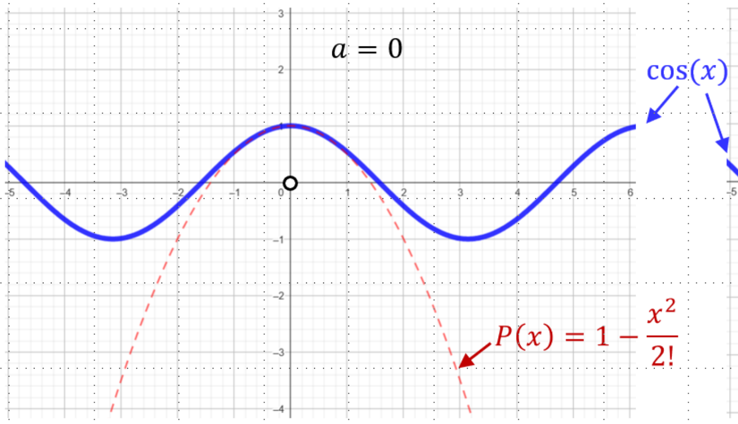
$cos^{(5)}(0) = -\sin(0) = 0$

$cos^{(6)}(0) = -\cos(0) = -1$

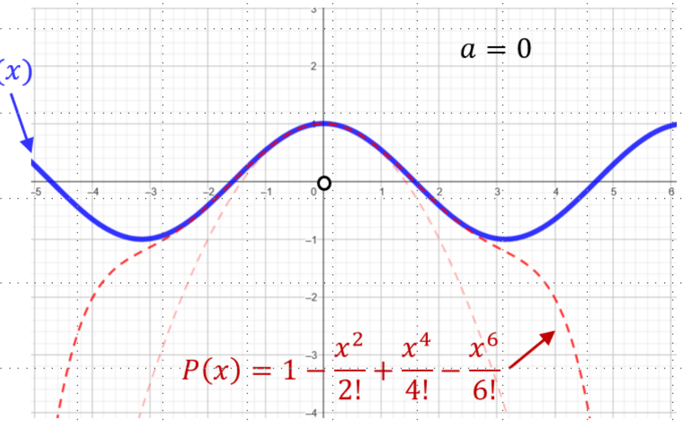
$cos^{(7)}(0) = \sin(0) = 0$

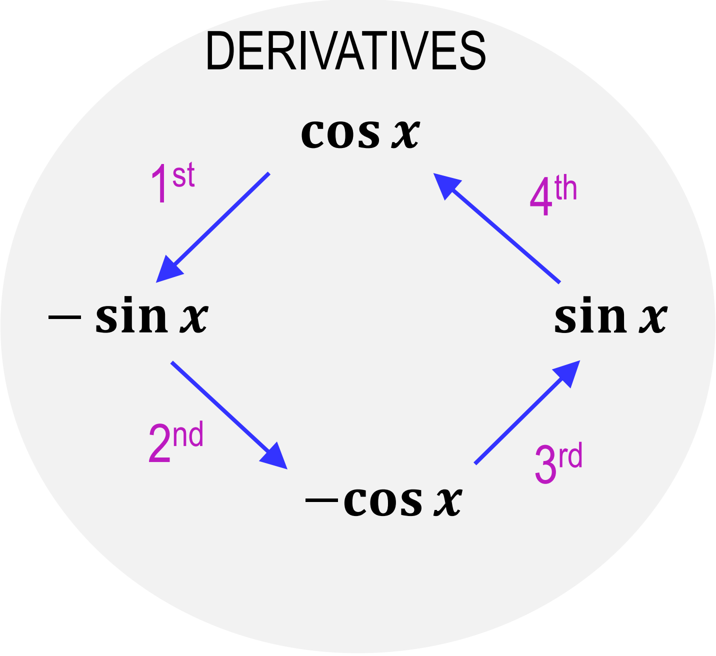
$cos^{(8)}(0) = \cos(0) = 1\ldots..$

EXAMPLE 1



EXAMPLE 2





$$(sin)' = cos$$

$$(cos)' = -sin$$

Taylor polynomials (2nd example)

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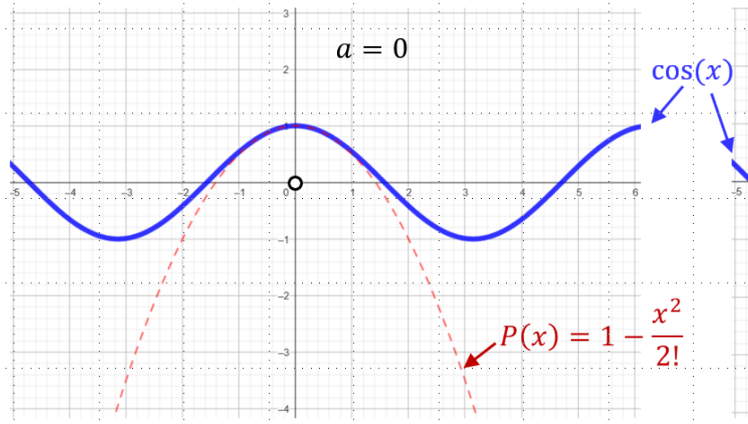
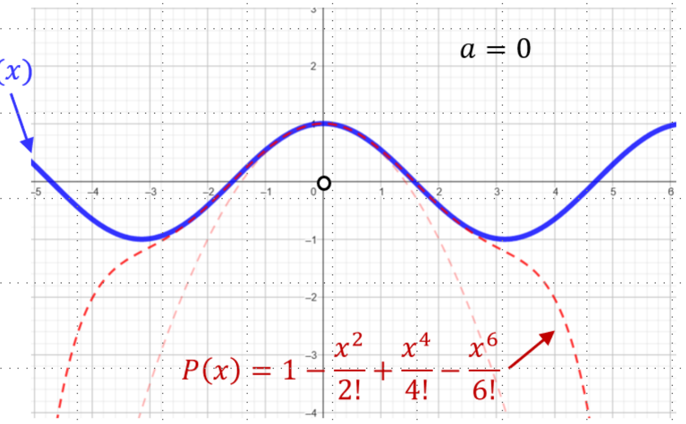


EXAMPLE 2



$$P_{2,0}(x) = 1 - \frac{x^2}{2!} \qquad (n = 2, a = 0)$$

$$P_{6,0}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \qquad (n = 6, a = 0)$$

$\cos x \approx P_{2,0}(x), \qquad x \in (-1, 1)$

$\cos x \approx P_{6,0}(x), \qquad x \in (-2.5, 2.5)$

$$P_{n,a}(x) = f(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots \frac{f^{(n)}(a)}{n!}(x-a)^n$$

It's your turn!

$$f(x) = \sin\left(x + \frac{\pi}{4}\right) \qquad (a = 0)$$

Which of the following represents $\boldsymbol{P_{3,0}(x)}$ for the above function?

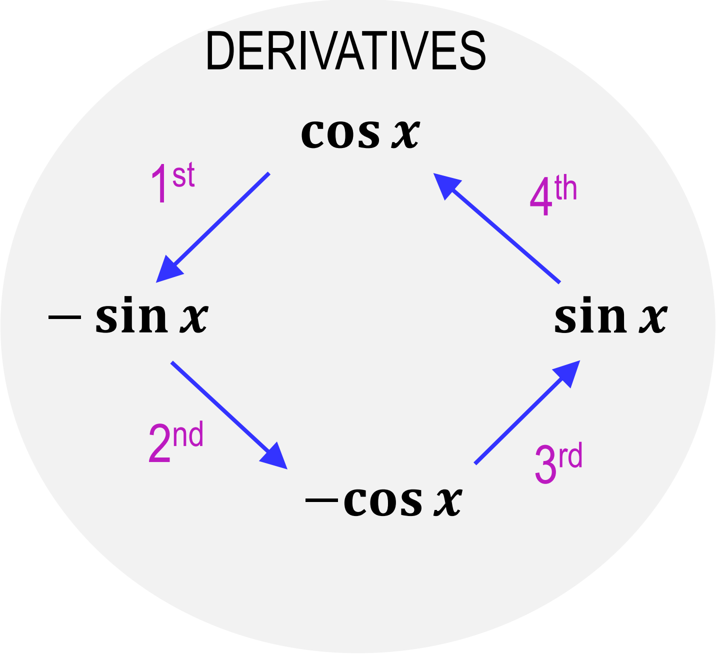
A). $x - \frac{x^3}{6\sqrt{2}}$

B). $\frac{1}{\sqrt{2}} + x\sqrt{2} - 2x^2 - \frac{x^3}{6\sqrt{3}}$

C). $\frac{1}{\sqrt{2}} + \frac{x}{\sqrt{2}} - \frac{x^2}{2\sqrt{2}} - \frac{x^3}{6\sqrt{2}}$

D). $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}x^2}{4} - \frac{x^3}{6\sqrt{2}}$

E). None of the above



$$(\sin)' = \cos$$

$$(\cos)' = -\sin$$

$$P_{n,a}(x) = f(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor's Theorem

Theorem 1: Let $n \in \mathbb{N}$ be a fixed natural number.
If $f: I \subseteq \mathbb{R} \to \mathbb{R}$ has $(n+1)$ continuous derivatives on an open interval $J \subseteq I$ that contains the point a, then for each $x \in J$

$$f(x) = P_{n,a}(x) + R_{n,a}(x)$$

with

$$R_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$ for some number c between a and x.

Taylor's Theorem gives us an explicit expression for the **remainder** (or **error**) **term** $R_{n,a}(x)$

FURTHER REMARKS

Taylor's Theorem

$$P_{n,a}(x) = f(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots \frac{f^{(n)}(a)}{n!}(x-a)^n$$

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controls the **error/accuracy** of the approximation

$$f(x) \approx P_{n,a}(x)$$

$$f(x) = f(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some $c \in (a, x)$ or $c \in (x, a)$

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for some $c \in (a, x)$ or $c \in (x, a)$

An important remark:

Taylor's Theorem does **not** tell us how to find the point c, only that it exists. Therefore, to estimate the size of the remainder term $R_{n,a}(x)$ we must **estimate** how large $\left|f^{(n+1)}(c)\right|$ could be *without knowing the value of* c.

Example for estimating the accuracy….

Use $P_{1,0}(x)$ to estimate $\sin(0.01)$ and show that the **error** in using this approximation is less than 10^{-4}

$f(x) = \sin(x)$

$a = 0.01$

Solution:

We know that $f(0) = \sin(0) = 0$ and $f'(0) = \cos(0) = 1,$ so

$$P_{1,0}(x) = f(0) + f'(0)x = x$$

Therefore, $\sin(0.01) \approx P_{1,0}(0.01) = 0.01 \implies \sin(0.01) \approx 0.01$

In fact, **Taylor's Theorem** gives: $\sin(0.01) = P_{0,1}(0.01) + R_{1,0}(0.01)$

and it also guarantees that there exists some $c \in (0,\ 0.01)$ such that:

$$|R_{1,0}(0.01)| = \left|\frac{f^{(2)}(c)}{2!}(0.01-0)^2\right| = \left|\frac{-\sin(c)}{2!}(0.01)^2\right|$$

$$\leq \frac{(0.01)^2}{2} < (0.01)^2 = 10^{-4}$$

$|-\sin(c)| \leq 1$