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Regular & Singular Perturbations

1^{st} Year Calculus: Maclaurin series

$f = f(x)$ a real-valued function

For x close to zero:

$$f(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \cdots + \frac{1}{n!}f^{(n)}(0)x^n + \cdots$$

NOTATIONS:

$$f'(0) = \frac{df}{dx}(0) \qquad f''(0) = \frac{d^2f}{dx^2}(0) \qquad \ldots \qquad f^{(n)}(0) = \frac{d^nf}{dx^n}(0)$$

1st Year Calculus: Maclaurin series

E.G.:

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \qquad \text{valid for all } x \in \mathbb{R}$$

For a **fixed** value of x ($x = a$, say), the RHS can be thought of as an approximation of $f(a)$ if we take a finite number of terms:

$$\cosh(a) = 1 + \frac{a^2}{2!} + \frac{a^4}{4!} + \frac{a^6}{6!} + \cdots$$

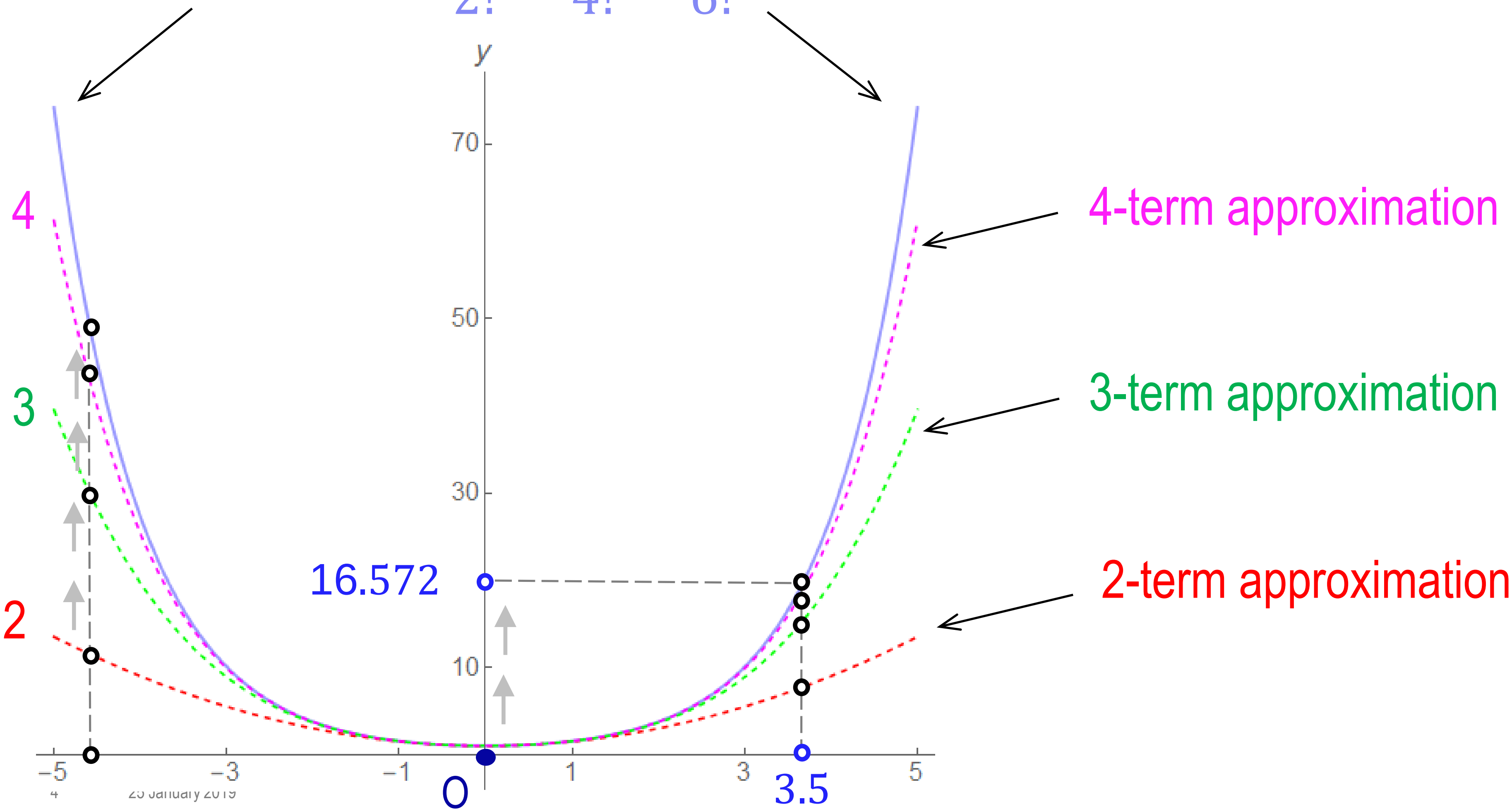
The sum gets arbitrarily close to $f(a)$ as we take more terms: called **convergent series**

Illustration

$a = 3.5$

$\cosh(a) = 16.572$

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$



Asymptotic expansions

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \Longrightarrow \quad \sin \varepsilon = \varepsilon - \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{5!} - \frac{\varepsilon^7}{7!} + \cdots$$

ε small positive number

If we stop after one term there is an **error**, which is small if ε is small:

$$\sin \varepsilon = \varepsilon \underbrace{- \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{5!} - \frac{\varepsilon^7}{7!} + \cdots}_{R_1 \text{ (error term)}} \quad \Longrightarrow \quad \frac{R_1}{\varepsilon} = \underbrace{-\frac{\varepsilon^2}{3!} + \frac{\varepsilon^4}{5!} - \frac{\varepsilon^6}{7!} + \cdots}_{\text{tends to zero as } \varepsilon \to 0}$$

To summarise:

If we stop after one term there is an **error**,
which is small if ε is small. Thus,

$$\sin\varepsilon = \varepsilon + R_1,$$

where $R_1 \ll \varepsilon$ as ε gets small

$(R_1/\varepsilon \to 0, \text{ as } \varepsilon \to 0)$

This represents a (leading-order) **asymptotic approximation**.

Write $\sin\varepsilon \sim \varepsilon$ read as "$\sin\varepsilon$ is **asymptotically equal to** $\boldsymbol{\varepsilon}$ as ε gets small"

Higher-order approximations (I)

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$$\sin\varepsilon \sim \varepsilon - \frac{\varepsilon^3}{3!}$$

means

$$\sin\varepsilon = \varepsilon - {\varepsilon^3}/{3!} + R_3,$$

with R_3 negligible compared to ε^3

$$(R_3/\varepsilon^3 \to 0, \quad \text{as} \quad \varepsilon \to 0)$$

$$\sin\varepsilon = \varepsilon - \frac{\varepsilon^3}{3!} + \underbrace{\frac{\varepsilon^5}{5!} - \frac{\varepsilon^7}{7!} + \cdots}_{R_3} \quad \Longrightarrow \quad \frac{R_3}{\varepsilon^3} = \underbrace{\frac{\varepsilon^2}{5!} - \frac{\varepsilon^4}{7!} + \cdots}_{\text{tends to zero as } \varepsilon \to 0}$$

Higher-order approximations (II)

Higher-order approximations:

$$\sin\varepsilon \sim \varepsilon - \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{5!}$$

means $\sin\varepsilon = \varepsilon - \varepsilon^3/3! + \varepsilon^5/5! + R_5$,

with R_5 negligible compared to ε^5

$(R_5/\varepsilon^5 \to 0, \text{ as } \varepsilon \to 0)$

$$\sin\varepsilon = \varepsilon - \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{5!} \underbrace{- \frac{\varepsilon^7}{7!} + \cdots}_{R_5} \implies \frac{R_5}{\varepsilon^5} = \underbrace{-\frac{\varepsilon^2}{7!} + \cdots}_{\text{tends to zero as } \varepsilon \to 0}$$

Summary

$$\sin\varepsilon \sim \varepsilon - \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{5!} - \frac{\varepsilon^7}{7!} + \cdots$$

is called an **Asymptotic Expansion**

Convergent Series: Get accuracy by taking more and more terms (ε fixed)

Asymptotic Expansion: Get accuracy by taking $\varepsilon \to 0$ with fixed number of terms

OBS.

- Asymptotic Expansions are **not** always convergent.
- Maclaurin series are asymptotic expansions (**the converse is not true**).

Numerical illustration

$$\sin\varepsilon \sim \varepsilon - \frac{\varepsilon^3}{6} \quad \Longleftrightarrow \quad \sin\varepsilon = \varepsilon - \frac{\varepsilon^3}{3!} + \text{small error}$$

ε	ε	$\varepsilon - \varepsilon^3/6$	$\sin\varepsilon$
$\pi/4$	0.785398	0.704653	0.707107
$\pi/8$	0.392699	0.382606	0.382683
$\pi/16$	0.19635	0.195088	0.19509
$\pi/32$	0.0981748	0.098017	0.098017

Other important points

NOT all functions admit a Maclaurin expansion

Regular functions = those that have a such an expansion

Singular functions = those that do **not** have a Maclaurin expansion

↓

E.G.:

$$f(x) = \frac{1}{x}\,; \quad g(x) = e^{1/x}; \quad h(x) = \frac{x^2+1}{x(x-3)}$$

Regular approximations

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Consider the following quadratic depending on a parameter:

$$x^2 - 2x + \varepsilon = 0$$

(ε small positive number)

Solve using the usual formula, to get

$$x_1 = x_1(\varepsilon) \equiv 1 - \sqrt{1-\varepsilon}$$

$$x_2 = x_2(\varepsilon) \equiv 1 + \sqrt{1-\varepsilon}$$

regular functions

Maclaurin expansion

$$x_1(\varepsilon) \sim \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^3 + \cdots$$

Maclaurin expansion

$$x_2(\varepsilon) \sim 2 - \frac{1}{2}\varepsilon + \cdots$$

Regular approximations

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$$x_1(\varepsilon) = x_1(0) + x_1'(0)\varepsilon + \frac{1}{2}x_1''(0)\varepsilon^2 + \cdots$$

$$x_2(\varepsilon) = x_2(0) + x_2'(0)\varepsilon + \cdots$$

can calculate the derivatives using **implicit differentiation**

Set $\varepsilon = 0$ in $\quad x^2(\varepsilon) - 2x(\varepsilon) + \varepsilon = 0 \Longrightarrow x^2(0) - 2x(0) = 0$

$\nearrow x_1(0) = 0$

$\searrow x_2(0) = 2$

Differentiate w.r.t ε:

$$x^2(\varepsilon) - 2x(\varepsilon) + \varepsilon = 0 \quad \Longrightarrow \quad 2x(\varepsilon)x'(\varepsilon) - 2x'(\varepsilon) + 1 = 0$$

$$\Longrightarrow x'(\varepsilon) = \frac{1}{2 - 2x(\varepsilon)}$$

$\nearrow x_1'(0) = 1/2$

$\searrow x_2'(0) = -1/2$

Regular approximations

To find $x''(0)$, **differentiate** again w.r.t. ε,

$$2x(\varepsilon)x'(\varepsilon) - 2x'(\varepsilon) + 1 = 0$$

$$\Longrightarrow \quad 2x'(\varepsilon)x'(\varepsilon) + 2x(\varepsilon)\underline{x''(\varepsilon)} - 2\underline{x''(\varepsilon)} + 0 = 0$$ [PRODUCT RULE]

$$\Longrightarrow \quad 2x'^2(\varepsilon) + x''(\varepsilon)[2x(\varepsilon) - 2] = 0$$ [COMBINING TERMS]

$$\Longrightarrow \quad x''(\varepsilon) = \frac{2x'^2(\varepsilon)}{2 - 2x(\varepsilon)}$$

$\nearrow$ $x_1''(0) = 1/4$

$\searrow$ $x_2''(0) = -1/4$

previous slide:

$x_1(0) = 0$

$x_2(0) = 2$

$x_1'(0) = 1/2$

$x_2'(0) = -1/2$

Results ($\varepsilon = 0.1$ and $\varepsilon = 0.01$)

We have found the following **approximations** (without solving the quadratic):

$$x_1 \sim \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + \cdots \qquad \text{and} \qquad x_2 \sim 2 - \frac{1}{2}\varepsilon + \cdots$$

ε	ONE-TERM APPROX.	TWO-TERM APPROX.	EXACT ROOTS
0.1	2.000	1.950	1.949
0.1	0.05	0.05125	0.05135
0.01	2.000	1.995	1.995
0.01	0.0050000	0.005013	0.005013

Singular perturbations

The previous **method fails** if we replace (say)

$$x^2 - 2x + \varepsilon = 0 \quad \text{by} \quad \varepsilon x^2 - 2x + 1 = 0$$

$$x_1 = x_1(\varepsilon) \equiv \frac{1 - \sqrt{1-\varepsilon}}{\varepsilon} \qquad x_2 = x_2(\varepsilon) \equiv \frac{1 + \sqrt{1-\varepsilon}}{\varepsilon}$$

ε	$x_1(\varepsilon)$
10^{-1}	0.5131670
10^{-2}	0.5012563
10^{-6}	0.5001000
10^{-7}	0.5000100

ε	$x_2(\varepsilon)$
10^{-1}	19.486833
10^{-2}	199.49874
10^{-6}	1999999.5
10^{-7}	19999999

Singular perturbations

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Re-scaling: $X = \varepsilon x \implies x = \frac{X}{\varepsilon}$

$$\varepsilon x^2 - 2x + 1 = 0 \implies \varepsilon\left(\frac{X}{\varepsilon}\right)^2 - 2\left(\frac{X}{\varepsilon}\right) + 1 = 0$$

$$\implies \frac{X^2}{\varepsilon} - \frac{2X}{\varepsilon} + 1 = 0 \implies X^2 - 2X + \varepsilon = 0$$

$$X \sim 2 - \frac{1}{2}\varepsilon + \cdots$$

$$X \sim \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + \cdots$$

$$x \sim \frac{2}{\varepsilon} - \frac{1}{2} + \cdots$$

$$x \sim \frac{1}{2} + \frac{1}{8}\varepsilon + \cdots$$