



The University of
Nottingham

Perturbation Methods



Problem

Find roots of cubic:

$$y = x^3 - 2x^2 + \varepsilon x \quad \varepsilon \text{ small number}$$

First thought: When $\varepsilon \rightarrow 0$, $y \approx y_0$ with

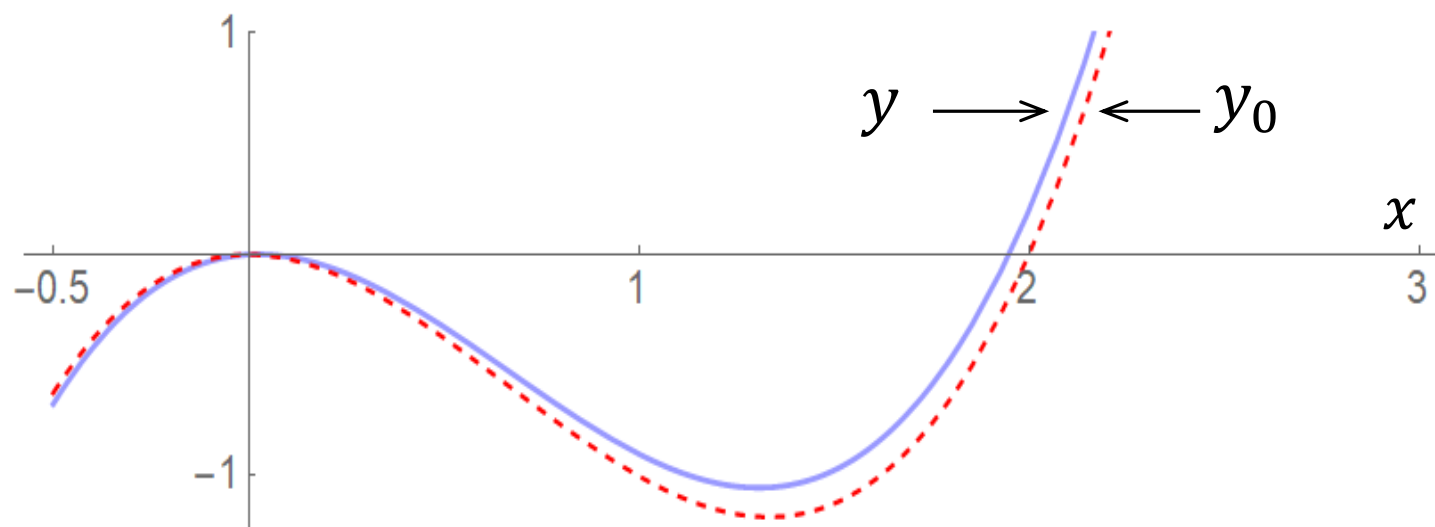
$$y_0 = x^3 - 2x^2$$

Roots of y_0 are at $x = 2, 0, 0$

$$x^2(x - 2) = 0$$



Graph of y and y_0 ($\varepsilon = 0.1$)





Improved estimates ($\varepsilon \neq 0$)

For root near $x = 2$:

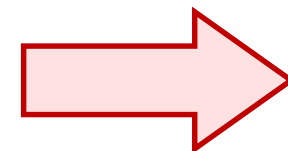
Put $x = 2 + a\varepsilon + b\varepsilon^2 + \dots$ so

$$(2 + a\varepsilon + b\varepsilon^2 + \dots)^3 - 2(2 + a\varepsilon + b\varepsilon^2 + \dots)^2 + \varepsilon(2 + a\varepsilon + b\varepsilon^2 + \dots) = 0$$

or

$$(8 + 12a\varepsilon + \dots) - (8 + 8a\varepsilon + \dots) + (2\varepsilon + \dots) = 0$$

$$\Rightarrow 4a\varepsilon + 2\varepsilon = 0 \Rightarrow 2\varepsilon(2a + 1) = 0$$





Improved estimates ($\varepsilon \neq 0$)

This gives $a = -1/2$ so improved value is

$$x = 2 - \frac{1}{2}\varepsilon$$

Other roots are near $x = 0$. **How near?**

Approximation $y \approx y_0$ is NOT very good

near $x = 0$: not accurate to discard term εx

$$y = x^3 - 2x^2 + \varepsilon x$$

$$y_0 = x^3 - 2x^2$$



The role of the “right” magnification

The grid contains 24 slides with the following titles:

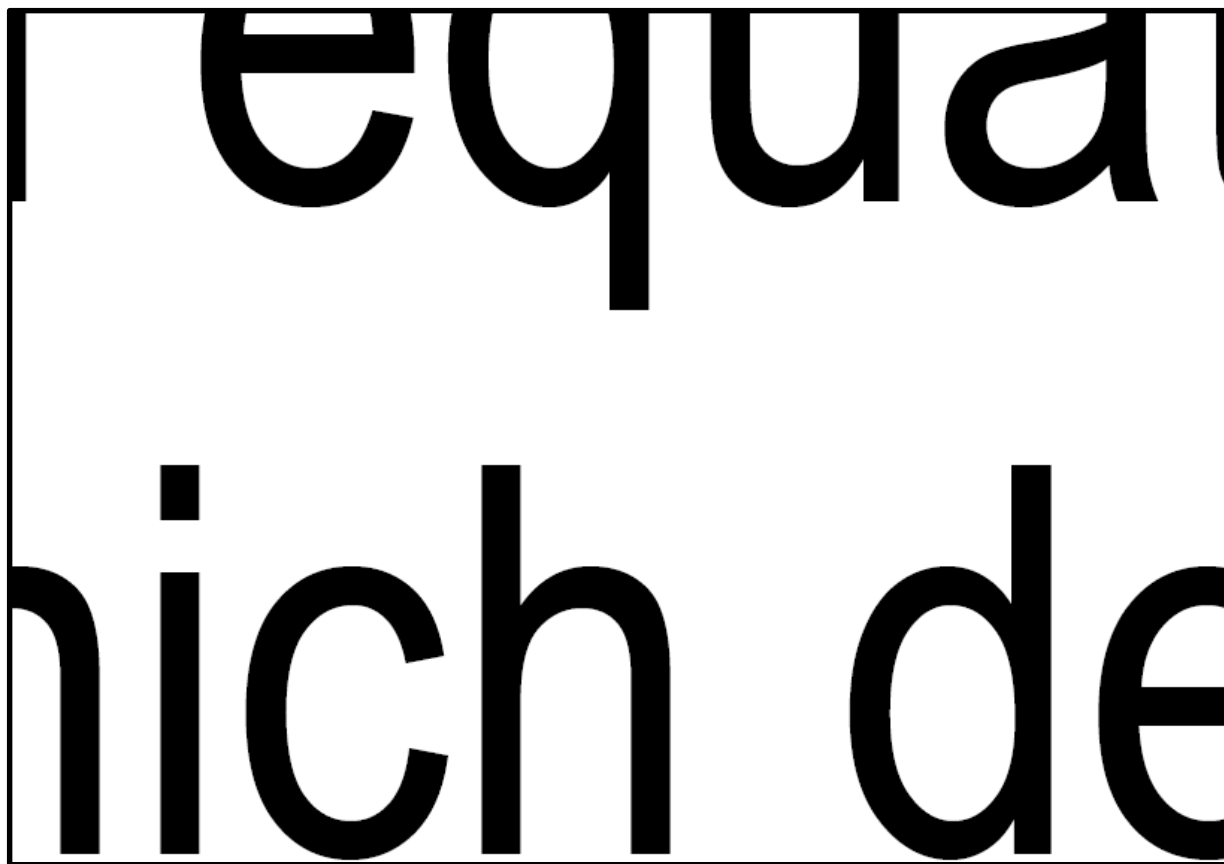
- Thin cylindrical shells subjected to vertical edge loads
- Magnification: pure bending of cylinders
- Problem outline
- Dependence of the magnification factor on λ/ℓ
- Asymptotic approximation
- Asymptotic approximation
- Agenda
- Magnification: pure bending of elastically cylinders
- Cylinder equations
- Dependence of the magnification factor on λ/ℓ
- Asymptotic approximation
- Asymptotic approximation
- Magnification
- Magnification: local buckling & pure bending
- ASDE
- Buckling modes (developed in the (λ, ℓ) plane)
- Comparisons for the local approximation
- Comparisons for the asymptotic approximations
- The buckling analysis
- Stress trajectories (shear)
- Simple observations
- Multiscale modelling & simulation of mechanical structures & materials
- Magnification: structural drilling
- Local & propagating buckles
- Shear buckling
- Bending buckling
- The transition between O.D. and S.D.
- Localized instabilities in order-elastic structures
- Magnification: offshore pipelines
- Magnification: pipeline installation
- Orientation of principal stresses
- Stress trajectories (bending)
- Euler-Bernoulli equations
- DEU and theory
- Loading experienced by pipelines
- Relevant works

too much information!



Blow-up of detail

previous slide
magnified
by a factor
of about 100



too little
information!



The root near zero

Key idea: Magnify picture to get better detail near $x = 0$.

Question: How much should we magnify in x and y directions?

$$y = x^3 - \underline{2x^2} + \underline{\varepsilon x}$$

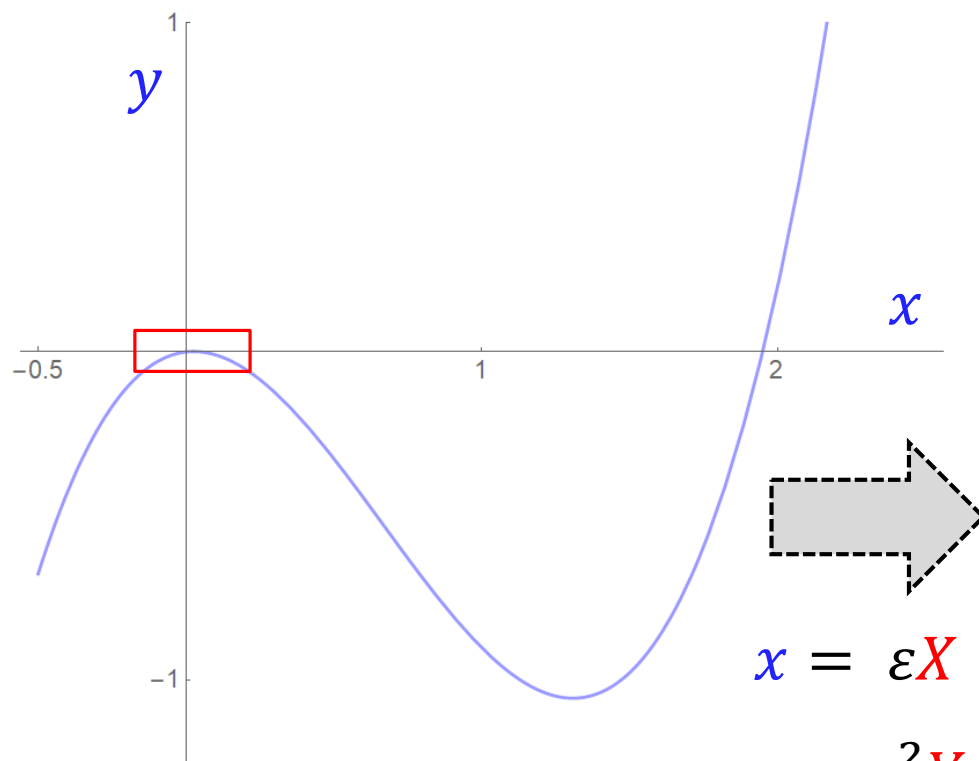
Answer: Balance the underlined terms, i.e. $\varepsilon x \approx x^2$
Similar magnitude when $x \approx \varepsilon$,
then y is of magnitude ε^2

Suggests scaling: $x = \varepsilon X$ and $y = \varepsilon^2 Y$



Graph of cubic & blow-up near $x = 0, y = 0$

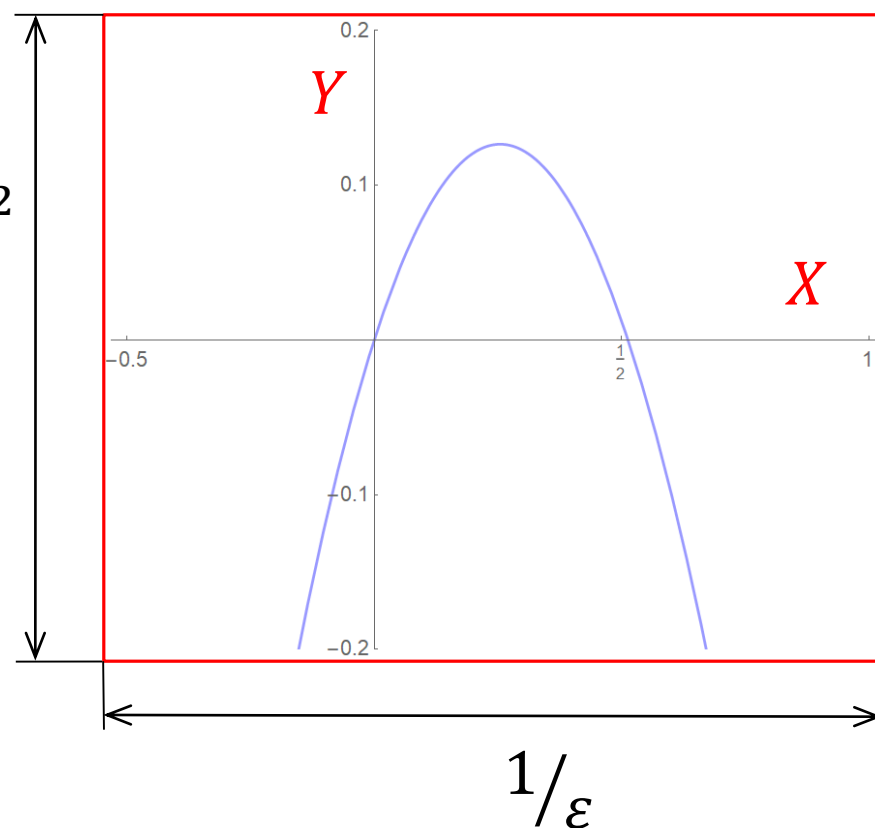
$$y = x^3 - 2x^2 + \varepsilon x$$



$$x = \varepsilon X$$

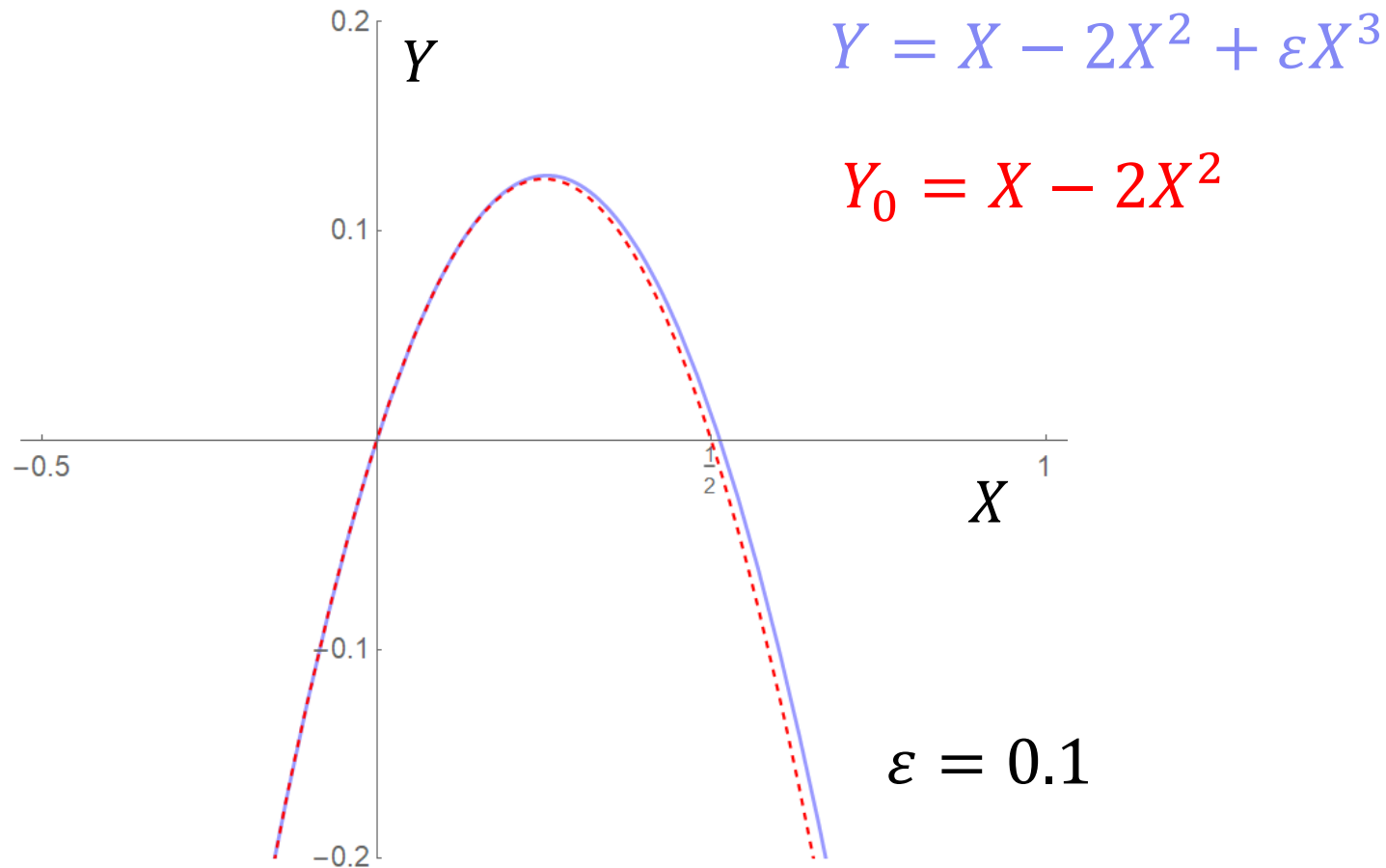
$$y = \varepsilon^2 Y$$

$$Y = \varepsilon X^3 - 2X^2 + X$$





The re-scaled cubic (near the origin)





Approximation of the root near zero

Roots of Y_0 are at $X = 0$ and $X = 1/2$ (i.e., $x = \varepsilon/2$)

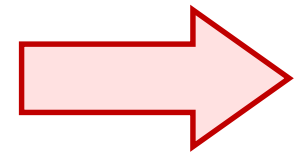
Improved estimate for root of Y near $X = 1/2$:

$$\text{Put } X = \frac{1}{2} + a\varepsilon + b\varepsilon^2 + \dots$$

Substitute into $X - 2X^2 + \varepsilon X^3 = 0$, to get

$$\left(\frac{1}{2} + a\varepsilon + \dots\right) - 2\left(\frac{1}{4} + a\varepsilon + \dots\right) + \left(\frac{\varepsilon}{8} + \dots\right) = 0$$

This gives $a = 1/8$, so $X = 1/2 + \varepsilon/8 + \dots$





Summary & results

Roots of the original cubic:

$$x = \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + \dots \quad \text{and} \quad x = 2 - \frac{1}{2}\varepsilon + \dots$$

ε	ONE-TERM APPROX.	TWO-TERM APPROX.	"EXACT"
0.1	2.000	1.950	1.949
0.1	0.05	0.05125	0.05135
0.01	2.000	1.995	1.995
0.01	0.0050000	0.005013	0.005013