The University of Nottingham

Perturbation Methods

Problem

Find roots of cubic:

$$y = x^3 - 2x^2 + \varepsilon x$$

ε small number

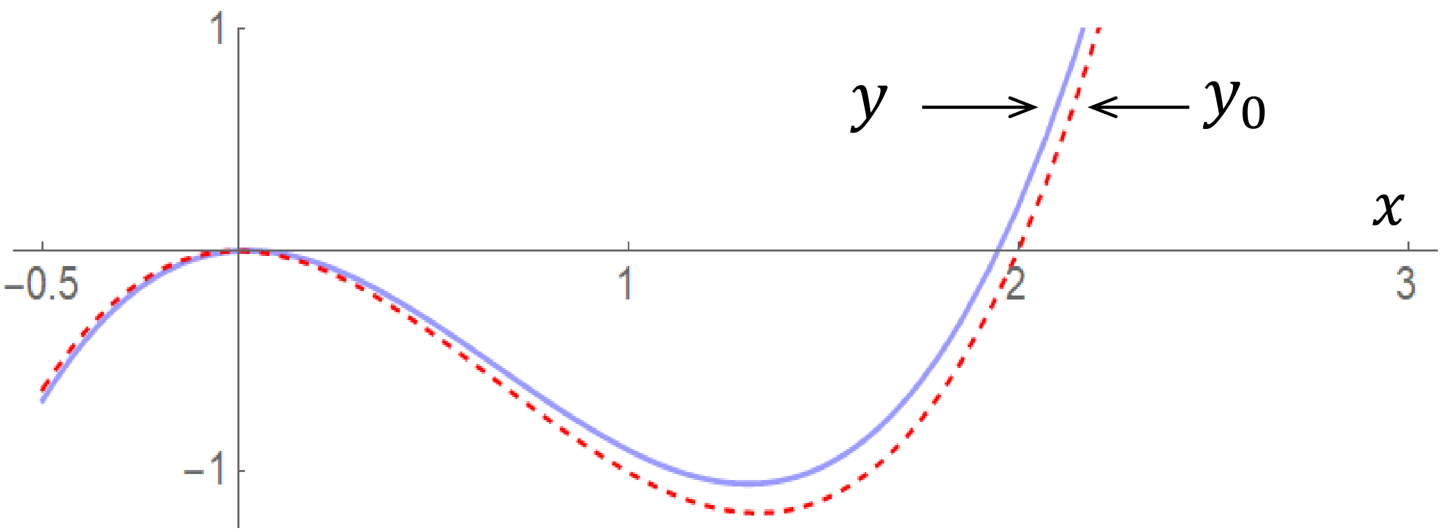
First thought: When $\varepsilon \to 0$, $y \approx y_0$ with

$$y_0 = x^3 - 2x^2$$

Roots of y_0 are at $x = 2,\ 0,\ 0$

$$x^2(x-2) = 0$$

Graph of y and y_0 ($\varepsilon = 0.1$)



Improved estimates $(\varepsilon \neq 0)$

For root near $x = 2$:

Put $x = 2 + a\varepsilon + b\varepsilon^2 + \cdots$ so

$$(2 + a\varepsilon + b\varepsilon^2 + \cdots)^3 - 2\,(2 + a\varepsilon + b\varepsilon^2 + \cdots)^2$$

$$+ \varepsilon(2 + a\varepsilon + b\varepsilon^2 + \cdots) = 0$$

or $$(8 + 12a\varepsilon + \cdots) - (8 + 8a\varepsilon + \cdots) + (2\varepsilon + \cdots) = 0$$

$$\Rightarrow \quad 4a\varepsilon + 2\varepsilon = 0 \quad \Rightarrow \quad 2\varepsilon(2a + 1) = 0$$

Improved estimates $(\varepsilon \neq 0)$

This gives $a = -1/2$ so improved value is

$$x = 2 - \frac{1}{2}\varepsilon$$

Other roots are near $x = 0$. **How near?**

Approximation $y \approx y_0$ is NOT very good

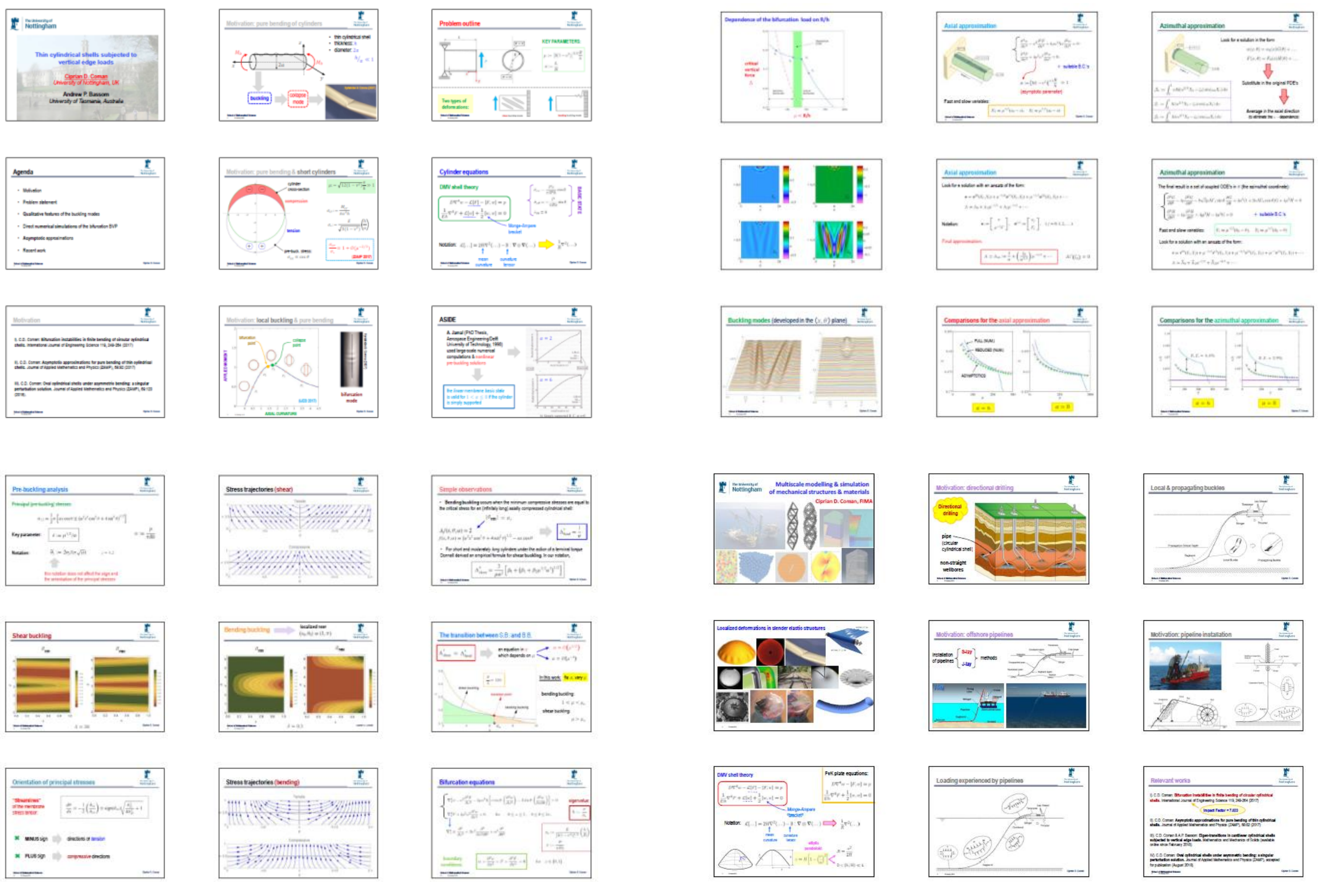
near $x = 0$: not accurate to discard term εx

$$y = x^3 - 2x^2 + \varepsilon x$$

$$y_0 = x^3 - 2x^2$$

Ciprian D. Coman

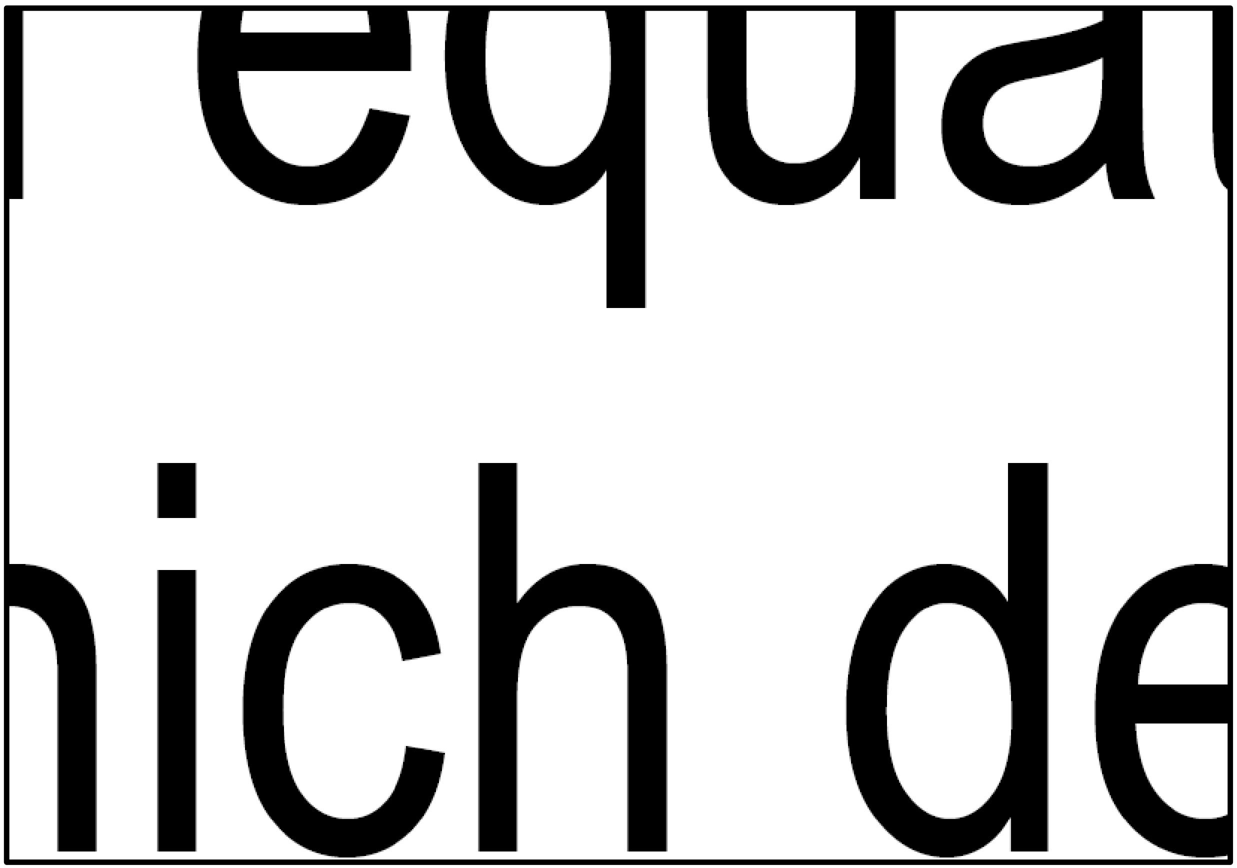
The role of the “right” magnification



too much information!

Blow-up of detail

previous slide magnified by a factor of about 100



too little information!

Ciprian D. Coman

The root near zero

Key idea: Magnify picture to get better detail near $x = 0$.

Question: How much should we magnify in x and y directions?

$$y = x^3 - \underline{2x^2} + \underline{\varepsilon x}$$

Answer:

Balance the underlined terms, i.e. $\varepsilon x \approx x^2$

Similar magnitude when $x \approx \varepsilon$,

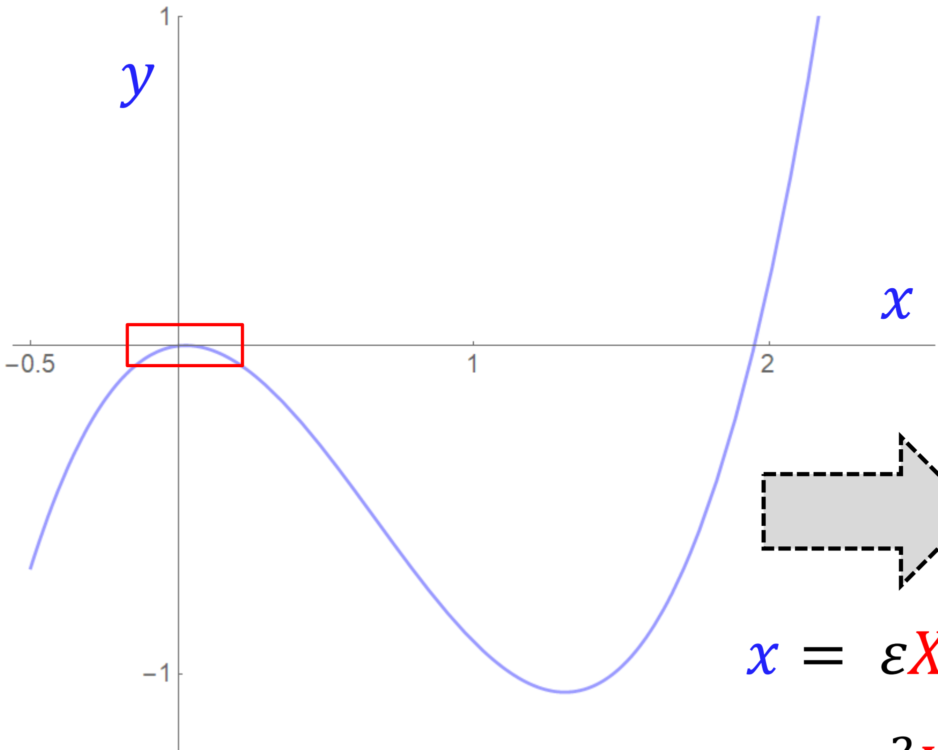
then y is of magnitude ε^2

Suggests scaling: $x = \varepsilon X$ and $y = \varepsilon^2 Y$

Graph of cubic & blow-up near $x = 0, y = 0$

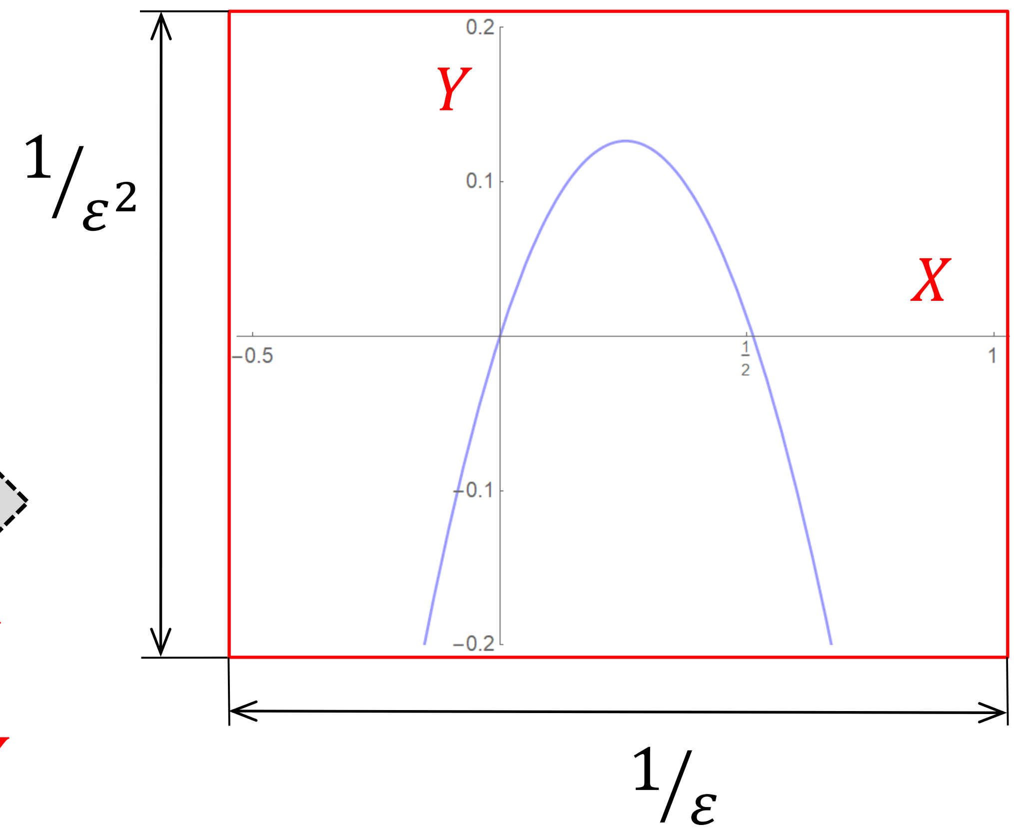
$$y = x^3 - 2x^2 + \varepsilon x$$

$$Y = \varepsilon X^3 - 2X^2 + X$$

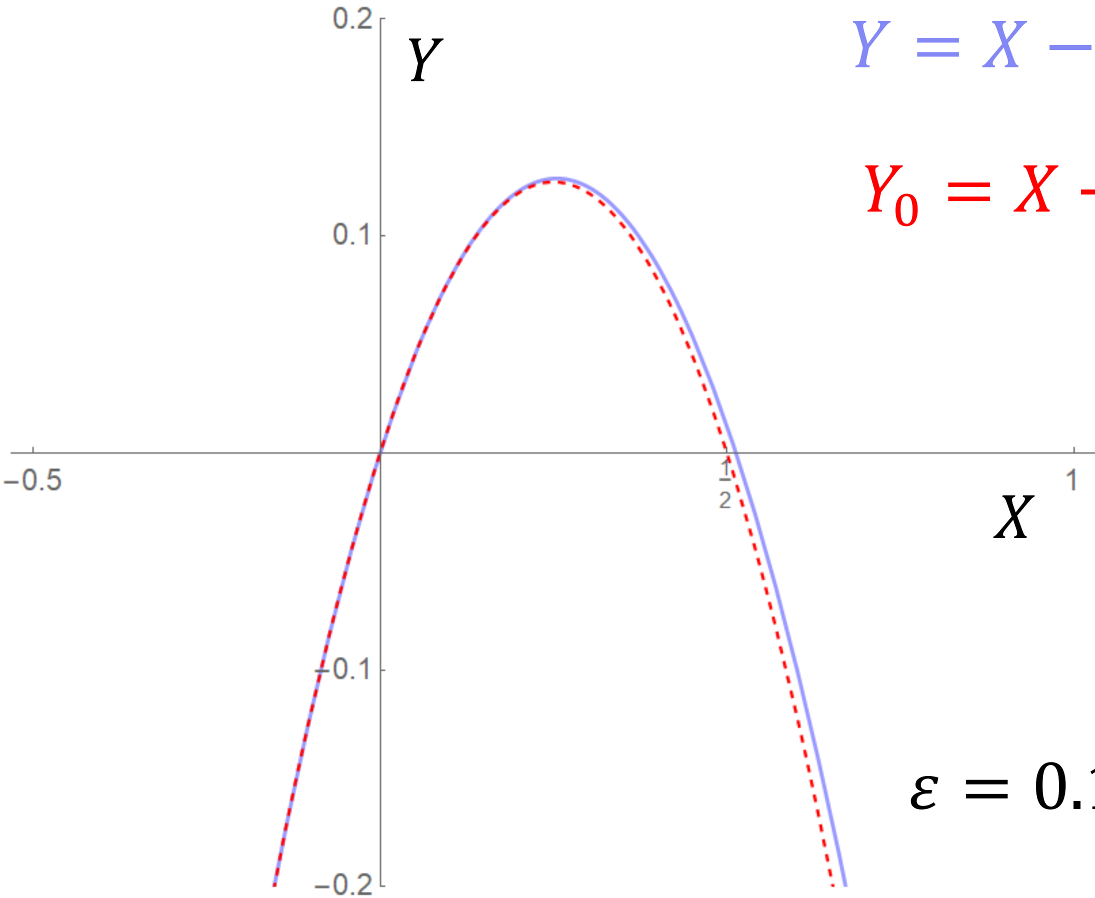


$$x = \varepsilon X$$

$$y = \varepsilon^2 Y$$



The re-scaled cubic (near the origin)



$$Y = X - 2X^2 + \varepsilon X^3$$

$$Y_0 = X - 2X^2$$

$$\varepsilon = 0.1$$

Approximation of the root near zero

Roots of Y_0 are at $X = 0$ and $X = {}^1/_2$ (i.e., $x = {}^\varepsilon/_2$)

Improved estimate for root of Y near $X = {}^1/_2$:

Put $X = \frac{1}{2} + a\varepsilon + b\varepsilon^2 + \cdots$

Substitute into $X - 2X^2 + \varepsilon X^3 = 0$, to get

$$\left({}^1/_2 + a\varepsilon + \cdots\right) - 2\left({}^1/_4 + a\varepsilon + \cdots\right) + \left({}^\varepsilon/_8 + \cdots\right) = 0$$

This gives $a = {}^1/_8$, so $X = {}^1/_2 + {}^\varepsilon/_8 + \cdots$

Summary & results

Roots of the original cubic:

$$x = \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + \cdots \quad \text{and} \quad x = 2 - \frac{1}{2}\varepsilon + \cdots$$

ε	ONE-TERM APPROX.	TWO-TERM APPROX.	"EXACT"
0.1	2.000	1.950	1.949
0.1	0.05	0.05125	0.05135
0.01	2.000	1.995	1.995
0.01	0.0050000	0.005013	0.005013