

# Integration by parts

# Refresher

Let  $h(x)$  be a given function,  $h : I \subset \mathbb{R} \rightarrow \mathbb{R}$

The **indefinite integral** of  $h(x)$  :

$$\int h(x) \, dx = H(x)$$

where

$$H'(x) = h(x)$$

(X)

**Observation:** If  $H(x)$  is replaced by  $H(x) + C$  (constant)  
the relation (X) above still holds true

○ if  $h_1(x) = h_2(x)$  then  $\int h_1(x) \, dx = \int h_2(x) \, dx$

**Properties:**

○  $\int [\alpha h(x)] \, dx = \alpha \left[ \int h(x) \, dx \right]$  ( $\alpha \in \mathbb{R}$ )

○  $\int [h_1(x) + h_2(x)] \, dx = \int h_1(x) \, dx + \int h_2(x) \, dx$

## Motivation

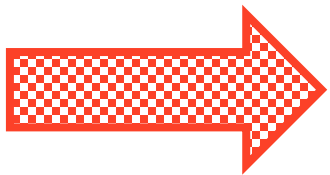
Want to calculate  $\int h(x) dx$  for

(i)  $h(x) = \cos(x)$ ;      (ii)  $h(x) = 2x$ ;      (iii)  $h(x) = e^x$

---

**Observation:** Note that  $h(x) = H'(x)$ , where

(i)  $H(x) = \sin(x)$ ;      (ii)  $H(x) = x^2$ ;      (iii)  $H(x) = e^x$



$$\int h(x) dx = \int H'(x) dx = H(x), \text{ etc}$$

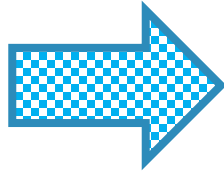
---

**Question:** What happens if  $h(x) = U(x)V'(x)$  ?

$$\int h(x) dx = \int U(x)V'(x) dx = ?$$

# The integration-by-parts formula

Product rule



$$\frac{d}{dx}[U(x)V(x)] = U(x)\frac{dV}{dx} + V(x)\frac{dU}{dx}$$

+

INTEGRATE:

$$U(x)V(x) = \int U(x)\frac{dV}{dx} + \int V(x)\frac{dU}{dx}$$

+

RE-ARRANGE:

$$\int U(x)\frac{dV}{dx} dx = U(x)V(x) - \int V(x)\frac{dU}{dx} dx$$

**Simplified  
form:**

$$\int U dV = UV - \int V dU$$

# A very simple example

$$\int U dV = UV - \int V dU$$

Want to calculate  $\int x e^x dx$  using the above formula

Need to identify/find  $U, V, dU, dV$

$$\int x e^x dx$$

$U \leftarrow$  (points to  $x$ )  
 $\rightarrow dV$  (points to  $e^x dx$ )

let  $U = x, \quad dV = e^x dx$   
then  $dU = dx, \quad V = e^x$

$$= x e^x - \int e^x dx$$

$$= x e^x - e^x + C \quad (C \in \mathbb{R})$$



ASIDE

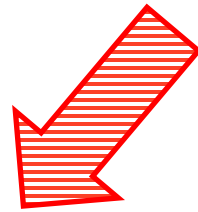
$$dU = U'(x) dx$$

$$V = \int dV$$

## General advice

$$\int U \, dV = UV - \int V \, dU$$

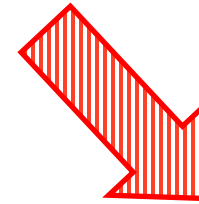
Look for a factor of the integrand that is *easily integrated* and include  $dx$  with that factor to make up  $dV$ . Then  $U$  is the remaining part of the integrand (sometimes might be necessary to take  $dV=dx$  only).



$$\text{integrand} = \text{polynomial} \times \left\{ \begin{array}{l} \text{exponential} \\ \text{sine} \\ \text{cosine} \end{array} \right\}$$

$$U = \text{polynomial}$$

$$dV = \text{the rest}$$



$$\text{integrand} = \left\{ \begin{array}{l} \text{log} \\ \text{inverse trig function} \\ \text{some other function} \\ \text{not readily integrable} \end{array} \right\}$$

$$U = \text{one of the above}$$

$$dV = \text{the rest}$$

# Examples

Want to calculate  $\int x^3 \log(x) dx$

$$\int U dV = UV - \int V dU$$

$$= \int \log(x) \cdot x^3 dx$$

$$= \frac{1}{4}x^4 \log(x) - \int \frac{1}{4}x^4 \cdot [\log(x)]' dx$$

$$= \frac{1}{4}x^4 \log(x) - \frac{1}{4} \int x^3 dx$$

$$= \frac{1}{4}x^4 \log(x) - \frac{x^4}{4 \cdot 4} + C$$

$$= \frac{1}{4}x^4 \left[ \log(x) - \frac{1}{4} \right] + C$$

let  $U = \log(x)$ ,  $dV = x^3 dx$   
then  $dU = [\log(x)]' dx$ ,  $V = \frac{1}{4}x^4$

$$[\log(x)]' = \frac{1}{x}$$

# Examples

Want to calculate  $\int x^2 \sin(x) dx$

$$\int \underline{x^2} \underline{\sin(x)} dx$$



$$\int U dV = UV - \int V dU$$

let  $U = \underline{x^2}$ ,  $dV = \underline{\sin(x) dx}$   
then  $dU = 2x dx$ ,  $V = -\cos(x)$



let  $U = \underline{x}$ ,  $dV = \underline{\cos(x) dx}$   
then  $dU = 1 \cdot dx$ ,  $V = \sin(x)$

$$= -x^2 \cos(x) + 2 \int \underline{x} \cdot \underline{\cos(x) dx}$$

$$= -x^2 \cos(x) + 2[x \sin(x) - \int \sin(x) dx]$$

$$= -x^2 \cos(x) + 2x \sin(x) + 2 \cos(x) + C$$

$$= (2 - x^2) \cos(x) + 2x \sin(x) + C$$