Integration by parts

Refresher

Let $h(x)$ be a given function, $h : I \subset \mathbb{R} \to \mathbb{R}$

The **indefinite integral** of $h(x)$:

$$\int h(x)\,\mathrm{d}x = H(x)$$

where

$$H'(x) = h(x) \qquad \text{(X)}$$

Observation: If $H(x)$ is replaced by $H(x) + C$ (constant)

the relation (X) above still holds true

Properties:

- if $h_1(x) = h_2(x)$ then $\int h_1(x)\,\mathrm{d}x = \int h_2(x)\,\mathrm{d}x$
- $\int [\alpha h(x)]\,\mathrm{d}x = \alpha \left[\int h(x)\,\mathrm{d}x\right] \quad (\alpha \in \mathbb{R})$
- $\int [h_1(x) + h_2(x)]\,\mathrm{d}x = \int h_1(x)\,\mathrm{d}x + \int h_2(x)\,\mathrm{d}x$

Motivation Want to calculate $\int h(x)\,\mathrm{d}x$ for

(i) $h(x) = \cos(x)$; (ii) $h(x) = 2x$; (iii) $h(x) = \mathrm{e}^x$

Observation: Note that $h(x) = H'(x)$, where

(i) $H(x) = \sin(x)$; (ii) $H(x) = x^2$; (iii) $H(x) = \mathrm{e}^x$

$$\Rightarrow \quad \int h(x)\,\mathrm{d}x = \int H'(x)\,\mathrm{d}x = H(x), \text{etc}$$

Question: **What happens if** $h(x) = U(x)V'(x)$ **?**

$$\int h(x)\,\mathrm{d}x = \int U(x)V'(x)\,\mathrm{d}x = ?$$

The integration-by-parts formula

Product rule ⇒

$$\frac{d}{dx}[U(x)V(x)] = U(x)\frac{dV}{dx} + V(x)\frac{dU}{dx}$$

+

INTEGRATE:

$$U(x)V(x) = \int U(x)\frac{dV}{dx} + \int V(x)\frac{dU}{dx}$$

+

RE-ARRANGE:

$$\int U(x)\frac{dV}{dx}\,\mathrm{d}x = U(x)V(x) - \int V(x)\frac{dU}{dx}\,\mathrm{d}x$$

Simplified form:

$$\int U\,\mathrm{d}V = UV - \int V\,\mathrm{d}U$$

A very simple example

$$\int U \, \mathrm{d}V = UV - \int V \, \mathrm{d}U$$

Want to calculate $\int x \, \mathrm{e}^x \, \mathrm{d}x$ using the above formula

Need to identify/find $U, \; V, \; \mathrm{d}U, \; \mathrm{d}V$

$$\int x \, \boxed{\mathrm{e}^x \, \mathrm{d}x}$$

U ← x; $\mathrm{e}^x \, \mathrm{d}x$ → dV

let $U = x \,, \quad \mathrm{d}V = \mathrm{e}^x \mathrm{d}x$

then $\mathrm{d}U = \mathrm{d}x \,, \quad V = \mathrm{e}^x$

$$= x\mathrm{e}^x - \int \mathrm{e}^x \, \mathrm{d}x$$

$$= x\mathrm{e}^x - \mathrm{e}^x + C \quad (C \in \mathbb{R})$$

ASIDE

$$dU = U'(x) dx$$

$$V = \int \mathrm{d}V$$

General advice

$$\int U\, \mathrm{d}V = UV - \int V\, \mathrm{d}U$$

Look for a factor of the integrand that is *easily integrated* and include **dx** with that factor to make up **dV**. Then **U** is the remaining part of the integrand (sometimes might be necessary to take **dV=dx** only).

$$\text{integrand} = \text{polynomial} \times \left\{ \begin{matrix} \text{exponential} \\ \text{sine} \\ \text{cosine} \end{matrix} \right\}$$

$$\boxed{\text{U} = \text{polynomial}}$$

$$\boxed{\text{dV} = \text{the rest}}$$

$$\text{integrand} = \left\{ \begin{matrix} \text{log} \\ \text{inverse trig function} \\ \text{some other function} \\ \text{not readily integrable} \end{matrix} \right\}$$

$$\boxed{\text{U} = \text{one of the above}}$$

$$\boxed{\text{dV} = \text{the rest}}$$

Examples

Want to calculate $\int x^3 \log(x)\, dx$

$$\int U\, dV = UV - \int V\, dU$$

let $U = \log(x)\,,\quad dV = x^3\, dx$

then $dU = [\log(x)]'\, dx\,,\quad V = \frac{1}{4}x^4$

$$= \int \log(x) \cdot x^3 dx$$

$$= \frac{1}{4}x^4 \log(x) - \int \frac{1}{4}x^4 \cdot [\log(x)]'\, dx$$

$$[\log(x)]' = \frac{1}{x}$$

$$= \frac{1}{4}x^4 \log(x) - \frac{1}{4}\int x^3\, dx$$

$$= \frac{1}{4}x^4 \log(x) - \frac{x^4}{4 \cdot 4} + C$$

$$= \frac{1}{4}x^4 \left[\log(x) - \frac{1}{4}\right] + C$$

Examples

$$\int U \, \mathrm{d}V = UV - \int V \, \mathrm{d}U$$

Want to calculate $\int x^2 \sin(x) \, \mathrm{d}x$

let $U = x^2$, $\mathrm{d}V = \sin(x) \, \mathrm{d}x$

then $\mathrm{d}U = 2x \, \mathrm{d}x$, $V = -\cos(x)$

$$\int x^2 \sin(x) \, \mathrm{d}x$$

let $U = x$, $\mathrm{d}V = \cos(x) \, \mathrm{d}x$

then $\mathrm{d}U = 1 \cdot \mathrm{d}x$, $V = \sin(x)$

$$= -x^2 \cos(x) + 2 \int x \cdot \cos(x) \, \mathrm{d}x$$

$$= -x^2 \cos(x) + 2\left[x \sin(x) - \int \sin(x) \, \mathrm{d}x\right]$$

$$= -x^2 \cos(x) + 2x \sin(x) + 2 \cos(x) + C$$

$$= (2 - x^2) \cos(x) + 2x \sin(x) + C$$