ODEs: existence & uniqueness

Ciprian D. Coman

Initial-Value Problem (IVP):

Solve: $y'(x) = f(x, y(x))$ ⟺ $y' = f(x, y)$ OR $\frac{dy}{dx} = f(x, y)$

Subject to: $y(x_0) = y_0$ (**initial condition**)

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QUESTIONS:

Given $f \equiv f(x, y)$ and $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}$:

a). Does a solution **exist**?

b). If a solution exists, is it **unique**?

Initial-Value Problem (IVP):

Solve: $y'(x) = f(x, y(x))$

Subject to: $y(x_0) = y_0$ **(initial condition)**

MOTIVATION: $xy' + y = x \qquad y(0) = 1$

NO SOLUTION: $xy' + y = x$ → (set $x = 0$) $(0)y'(0) + y(0) = 0$ → $1 = 0$ (!!)

Initial-Value Problem (IVP):

Solve: $y'(x) = f(x, y(x))$

Subject to: $y(x_0) = y_0$ **(initial condition)**

MOTIVATION:

$$y' = \frac{3}{2} y^{1/3} \qquad y(0) = 0$$

THREE DISTINCT SOLUTIONS:

$y(x) \equiv 0$ $\qquad$ $y(x) = -x^{3/2}$ $\qquad$ $y(x) = x^{3/2}$

Theorem 1 (G. Peano, 1890):

Let $A = \{(x,y) \in \mathbb{R} \times \mathbb{R} \colon |x - x_0| \le a, |y - y_0| \le b\}$

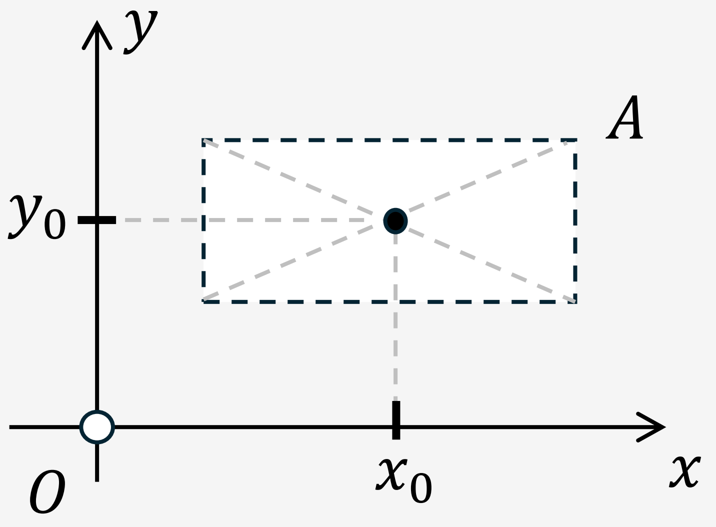
and suppose that $f\colon A \to \mathbb{R}$ is **continuous**, with $M \coloneqq \max\limits_{(x,y)\in A} |f(x,y)|$

Then the IVP

$$y'(x) = f(x, y(x)) \qquad y(x_0) = y_0$$

has **at least one solution** on $[x_0 - h, x_0 + h]$,

where $h := \min\left(a, \frac{b}{M}\right)$.



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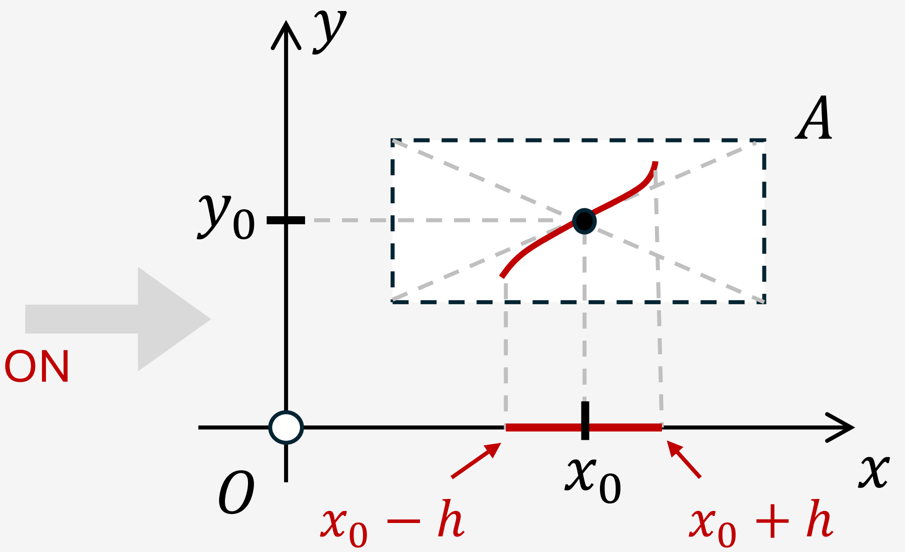
LOCAL SOLUTION



Observation:

While **Th.1** guarantees a solution for x close to x_0, this solution may not exist for all $x \in \mathbb{R}$:

E.G.

$$y' = 2xy^2 \qquad y(0) = 1$$

has the exact solution

$$y(x) = \frac{1}{1 - x^2}$$

← NOT defined at $x = \pm 1$

$$y: (-1,1) \to \mathbb{R}$$

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$$y' = 2xy^2 \qquad y(0) = 1 \qquad (*)$$

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$A = [-1,1] \times [0,2]$
$(a = 1, b = 1)$

$f(x,y) = 2xy^2$

$M := \max_{(x,y)\in A} |f(x,y)| = 8$

$h := \min(a, b/M) = \min(1,1/8\,) = 1/8$

Th. 1 predicts that a solution of (*) is defined on $\left[-\frac{1}{8}, \frac{1}{8}\right]$

Example:

$$y' = x^2 + y^2 \qquad y(0) = 0$$

Theorem 1 (G .Peano, 1890):

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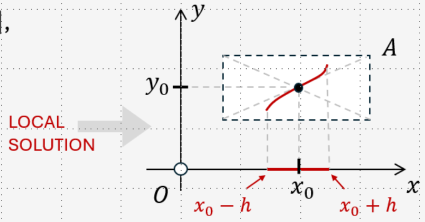

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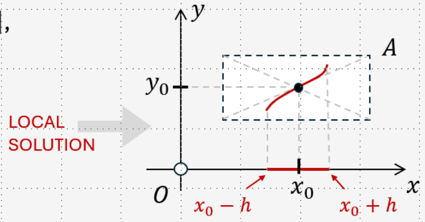

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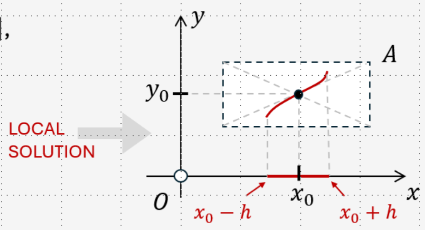

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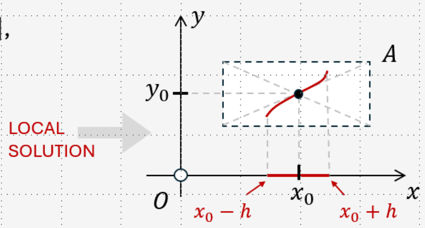

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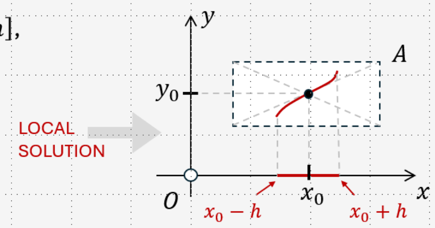

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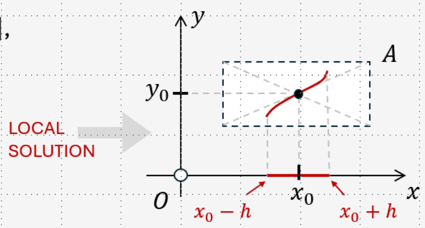

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STEP 4: Evaluate h in the formula:

$$h := \min(a, b/M) = \min(1, 1/2) = 0.5$$

$$M := \max_{(x,y)\in A} |f(x,y)| = 2$$

The predicted solution is defined on $[-0.5, 0.5]$

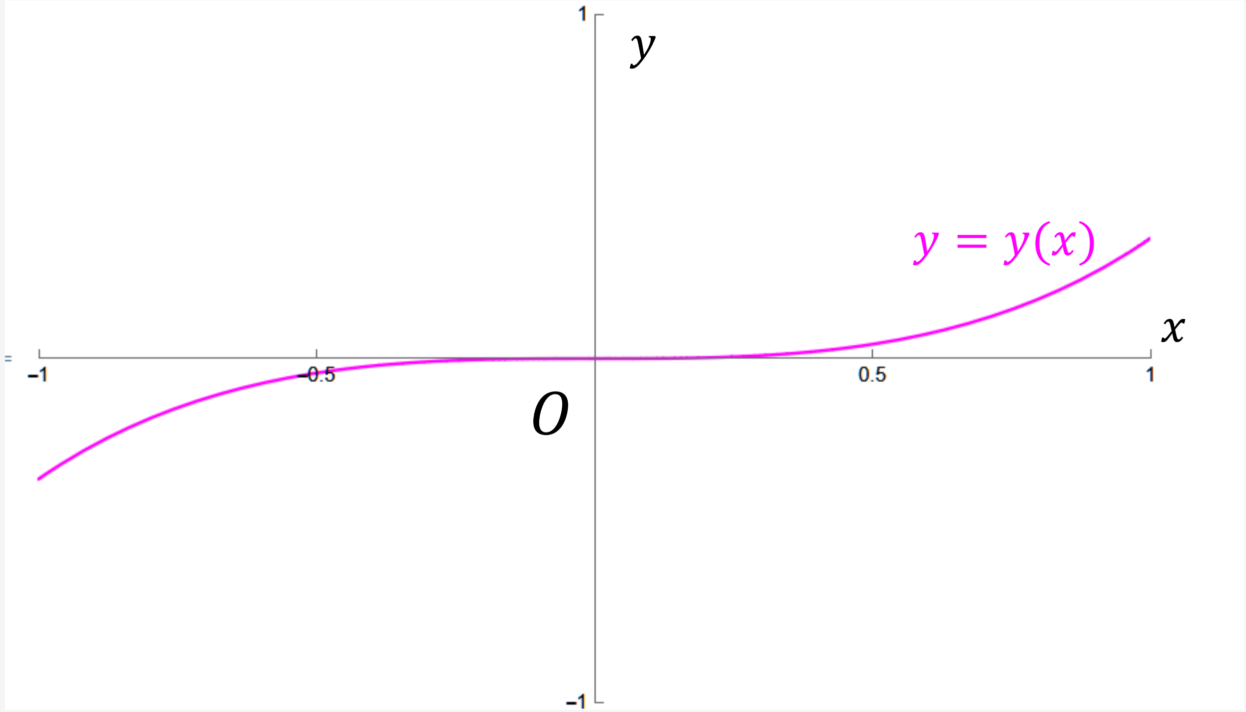
ASIDE:

$$y' = x^2 + y^2 \qquad y(0) = 0$$

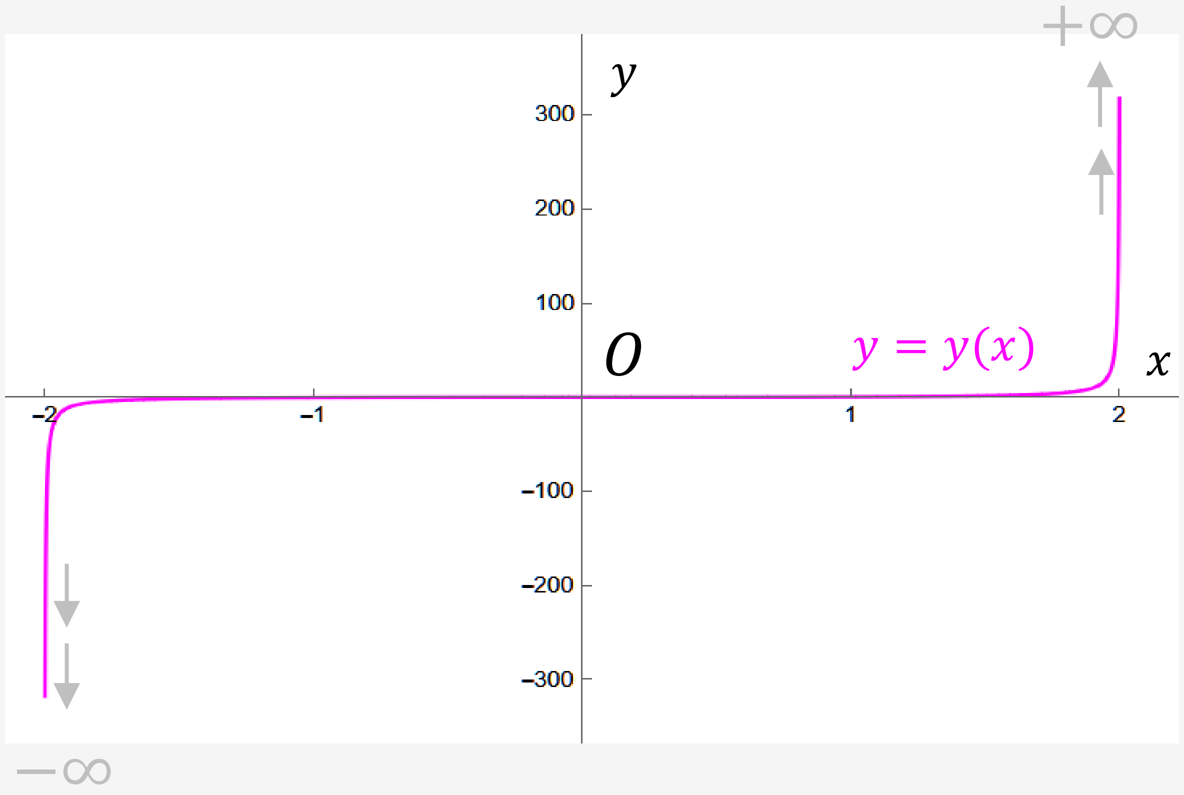
$$\lim_{x \to 2^-} y(x) = +\infty$$

$$\lim_{x \to -2^+} y(x) = -\infty$$

COMPUTER SOLUTIONS:



$-1 \le x \le 1$



$-2 \le x \le 2$

Theorem 2 (Picard-Lindelöf):

Suppose $f: \mathbb{R}^2 \to \mathbb{R}$ is **continuous** in some rectangle

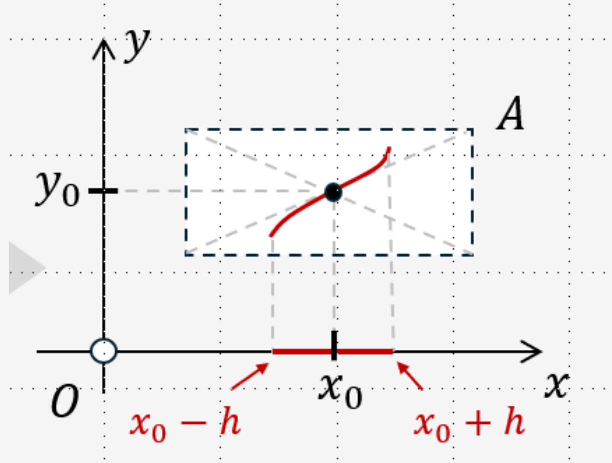
$$A = \{(x, y) \in \mathbb{R} \times \mathbb{R}: |x - x_0| \le a, |y - y_0| \le b\}$$

and that the **partial derivative** $f_y \equiv \dfrac{\partial f}{\partial y}(x, y)$ is also **continuous** there.

Then, there is an interval $[x_0 - h, x_0 + h]$ $(0 < h \le a)$ in which the IVP

$$y'(x) = f(x, y(x)) \qquad y(x_0) = y_0$$

has a UNIQUE solution



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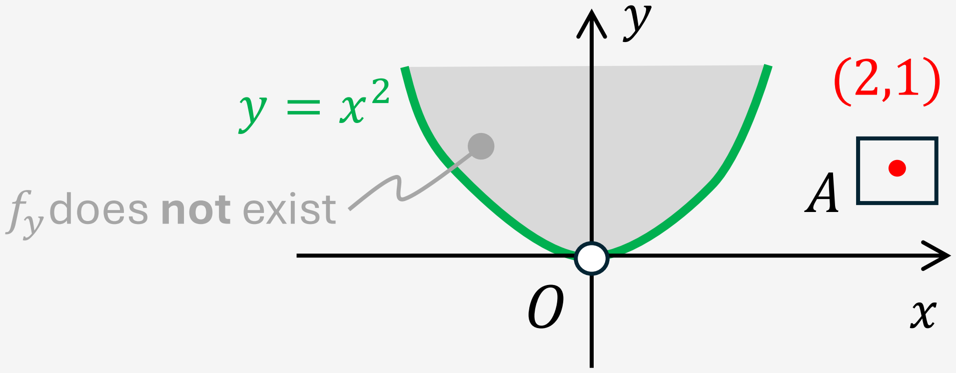
STEP 1: Identify $f(x, y) = 2x - 2\sqrt{x^2 - y}$ → continuous provided that $y \le x^2$

STEP 2: Calculate

$$f_y \equiv \frac{\partial f}{\partial y}(x, y) = \frac{1}{\sqrt{x^2 - y}}$$

continuous for $y < x^2$ → can find a *small* rectangle centred at $(x, y) = (2,1)$ such that both f and $\frac{\partial f}{\partial y}$ are continuous



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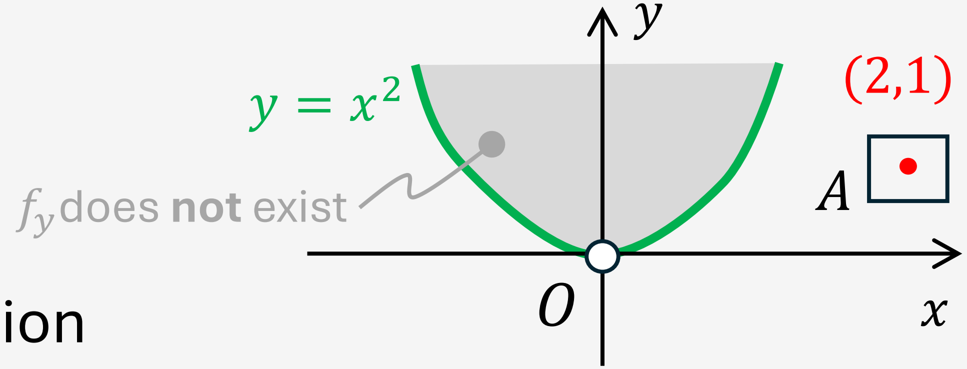

Conclusion: By **Th.2** our IVP does have a unique solution

Modified Example:

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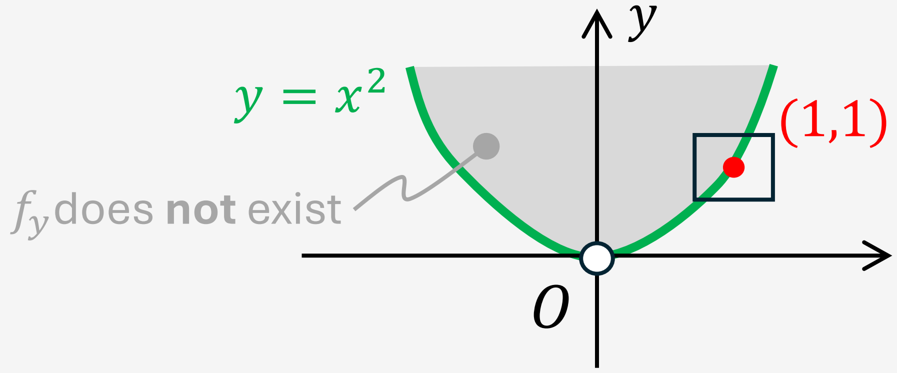
does NOT exist for $y = x^2$

there is **NO rectangle** centred at $(x,y) = (1,1)$ **such that f_y is continuous**



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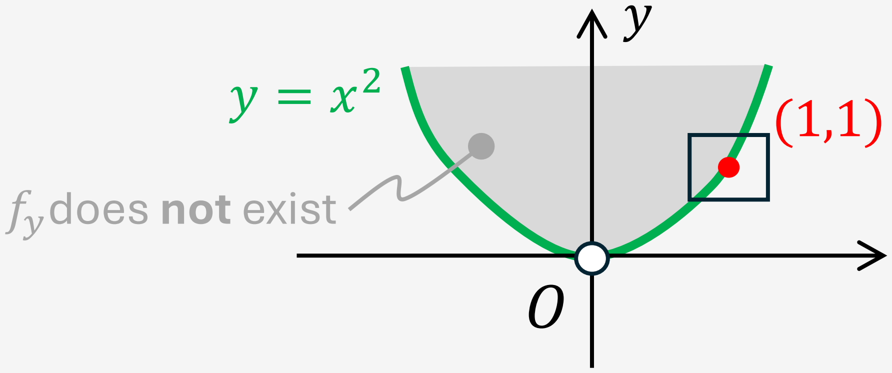
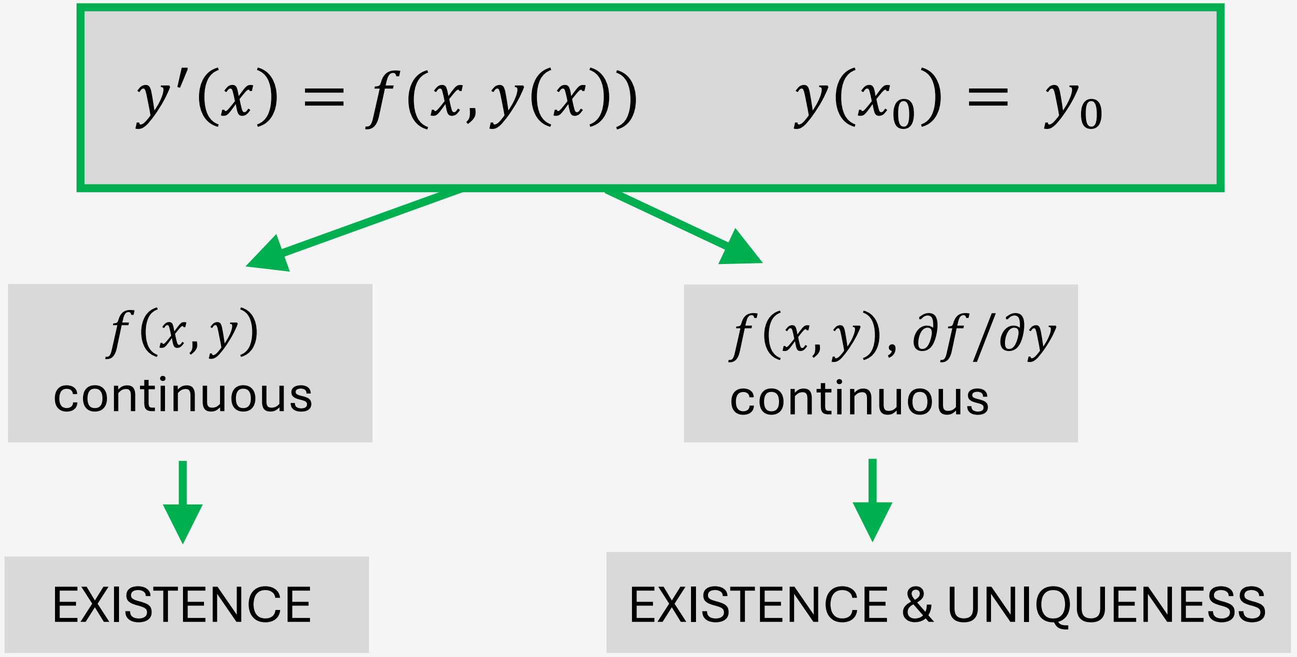


Conclusion: Th.2 is not applicable!!

$y: (-\infty, +\infty) \to \mathbb{R}$
$y(x) = x^2$

$y: [1, +\infty) \to \mathbb{R}$
$y(x) = 2x - 1$

Summary:



1. The Picard-Lindelöf theorem is an **existence theorem**, which means that if the right conditions are satisfied, we can find a solution, *but we are not told how to find it*.

2. The theorem also only establishes the existence of a **local solution**, in the vicinity of x_0.

3. The continuity of $f(x, y)$ and $\partial f/\partial y$ are **sufficient** conditions for the existence and uniqueness of solutions, *but they may not be necessary*.