University of HUDDERSFIELD

The Rules of Logarithms

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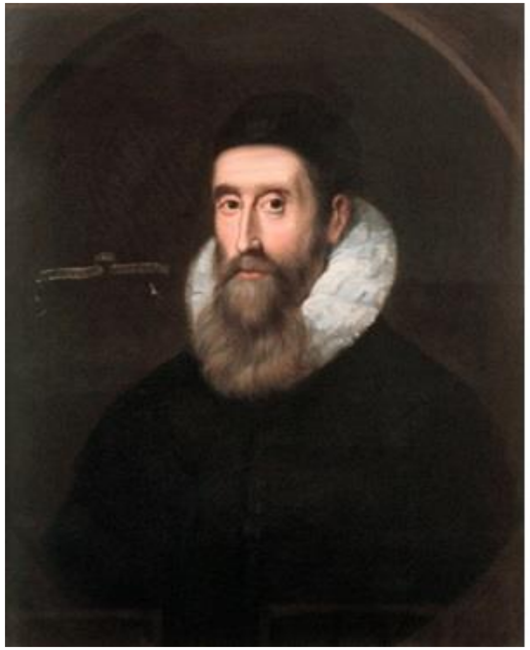
Outline

- Introduction
- **Exponentiation** (or raising a number to a power): $3^4, 12^2,$ etc
- What is a **logarithm** (or 'log') ?
- **Laws** of logarithms

Learning objectives: after completing this unit, you should be able to define logarithms, identify logarithm notation, and simplify expressions involving logarithms

The invention of logarithms



Merchiston Castle

John Napier (1614)

"Seeing there is nothing that is so troublesome to mathematical practice, nor that doth more molest and hinder calculators, than the multiplications, divisions, square and cubical extractions of great numbers…. I began therefore to consider in my mind by what certain and ready art I might remove those hindrances."

MIRIFICI

Logarithmorum
Canonis deſcriptio,

Ejuſque uſus, in utraque Trigonometria; ut etiam in omni Logiſtica Mathematica, Amplißimi, Facillimi, & expeditißimi explicatio.

Authore ac Inventore, IOANNE NEPERO, Barone Merchiſtonii, &c. Scoto.

EDINBURGI,
Ex officinâ ANDREÆ HART
Bibliopólæ, cIↃ. DC. XIV.

Some terminology

Exponentiation
(raising a number to a power)

$$5^3 = \underbrace{5 \times 5 \times 5}_{\text{3 times}} = 125$$

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$$5^3 = \underbrace{5 \times 5 \times 5}_{\text{3 times}} = 125$$

5^3 ← **exponent** (or **index**; or **power**)

↑ **base**

A **logarithm** is just the power that a number needs to be **raised to** produce a given value.

$$\log_5 125 = 3$$

What is a 'log'?

A **logarithm** is just the power that a number needs to be **raised to** produce a given value.

$\log_3 81 = 4$ → the same as $3^4 = 81$

$\log_2 32 = 5$ → the same as $2^5 = 32$

$\log_{10} 100 = 2$ → the same as $10^2 = 100$

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KEY RULE:

$$\log_B A = p$$

means the same as

$$B^p = A$$

$(B > 0, B \neq 1)$

Some special cases

$\log_B B = 1$ — the same as $B^1 = B$

$\log_B 1 = 0$ — the same as $B^0 = 1$

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Laws of logarithms

Motivation:

$\log_2 4 = 2$ $\quad$ $\log_2 16 = 4$ $\quad$ $\log_2 64 = 6$

$(2^2 = 4)$ $\quad$ $(2^4 = 16)$ $\quad$ $(2^6 = 64)$

Laws of logarithms

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$\log_a x = m$ and $\log_a y = n$	Take two logarithms with the same base
$x = a^m$ and $y = a^n$	Rewrite these expressions using powers
$xy = a^m \times a^n = a^{m+n}$	Multiply these powers
$\log_a xy = m + n = \log_a x + \log_a y$	Rewrite your result using logarithms

This result is one of the **laws of logarithms**.

You can use similar methods to prove two further laws.

- **The laws of logarithms:**
  - $\log_a x + \log_a y = \log_a xy$ **(the multiplication law)**
  - $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$ **(the division law)**
  - $\log_a (x^k) = k \log_a x$ **(the power law)**

Write as a single logarithm.

a $\log_3 6 + \log_3 7$ **b** $\log_2 15 - \log_2 3$ **c** $2\log_5 3 + 3\log_5 2$ **d** $\log_{10} 3 - 4\log_{10}\left(\frac{1}{2}\right)$

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a $\log_3 (6 \times 7)$ — Use the multiplication law.

$= \log_3 42$

b $\log_2 (15 \div 3)$

$= \log_2 5$ — Use the division law.

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c $2\log_5 3 = \log_5 (3^2) = \log_5 9$

$3\log_5 2 = \log_5 (2^3) = \log_5 8$

$\log_5 9 + \log_5 8 = \log_5 72$

First apply the power law to both parts of the expression.
Then use the multiplication law.

d $4\log_{10}\left(\frac{1}{2}\right) = \log_{10}\left(\frac{1}{2}\right)^4 = \log_{10}\left(\frac{1}{16}\right)$

$\log_{10} 3 - \log_{10}\left(\frac{1}{16}\right) = \log_{10}\left(3 \div \frac{1}{16}\right)$

$= \log_{10} 48$

Use the power law first.
Then use the division law.

Your Turn!

Write as a single logarithm, then simplify your answer **(without a calculator!)**

$$\log_8 25 + \log_8 10 - 3\log_8 5$$

The answer is:

A) 100

B) 1/3

C) $\log_8 13$

D) $\log_8 15$

E) None of the stated answers

- **The laws of logarithms:**
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The answer is:

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Solution: $log_8 25 + \ log_{10} 10 \ - \ log_8(5^3) = log_8(25 \times 10) - log_8 125$

$$= \ log_8\left(\frac{250}{125}\right) = log_8 2 = \frac{1}{3}$$

$$(2^3)^{1/3} = 2^{3\times\frac{1}{3}} = 2^1 = 2$$

Other laws (I):

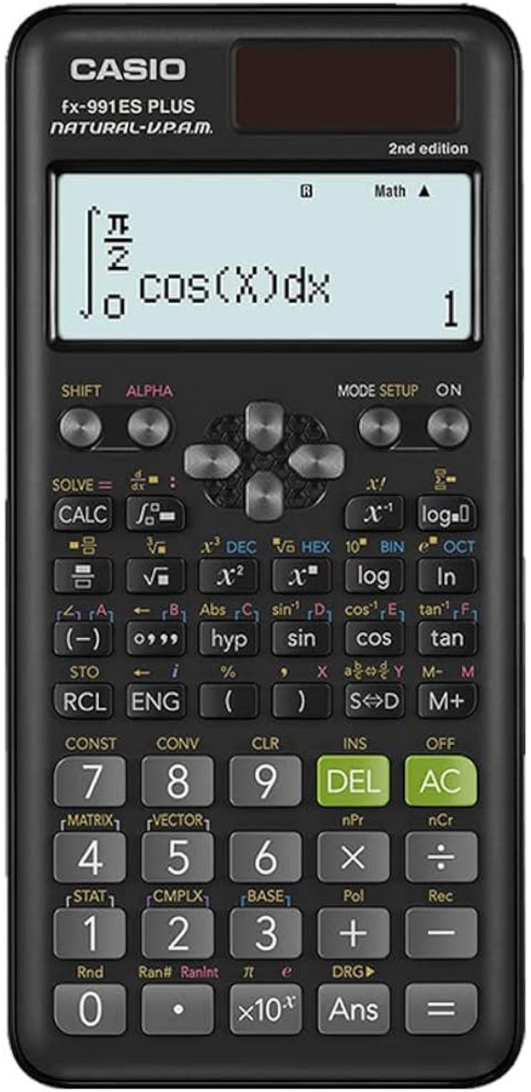
$$log_B A = \frac{1}{log_A B}$$

Examples:

$$log_{16}2 = \frac{1}{log_2 16} = \frac{1}{log_2(2^4)} = \frac{1}{4 \times log_2(2)} = \frac{1}{4}$$

$$log_{216}6 = \frac{1}{log_6 216} = \frac{1}{log_6(6 \times 36)} = \frac{1}{log_6(6^3)} = \frac{1}{3}$$

Other laws (II)



You can use your calculator to find logarithms of any base. Some calculators have a specific [log□□] key for this function. Most calculators also have separate buttons for logarithms to the base 10 (usually written as [log] and logarithms to the base e (usually written as [ln]).

$e = 2.718281828459045....$

(Euler's constant)

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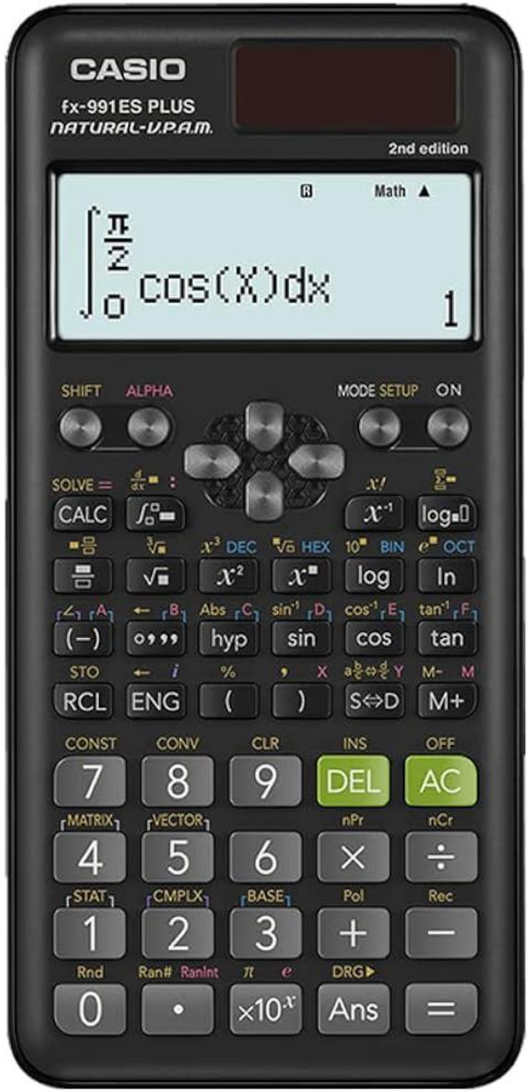

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$e = 2.718281828459045....$
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QUESTION: How can we find $log_3 40$ on a calculator that has only buttons for logs to the base 10 or natural logs?

Other laws (II)

$$log_B A = \frac{\ln A}{\ln B} = \frac{\log A}{\log B}$$

Example:

$$log_3 40 = \frac{\ln 40}{\ln 3} = \frac{3.68887945411}{1.09861228867} = 3.35776278143\ldots.$$

same answer!

$$log_3 40 = \frac{log_{10} 40}{log_{10} 3} = \frac{1.60205999133}{0.47712125472} = 3.35776278143\ \ldots.$$