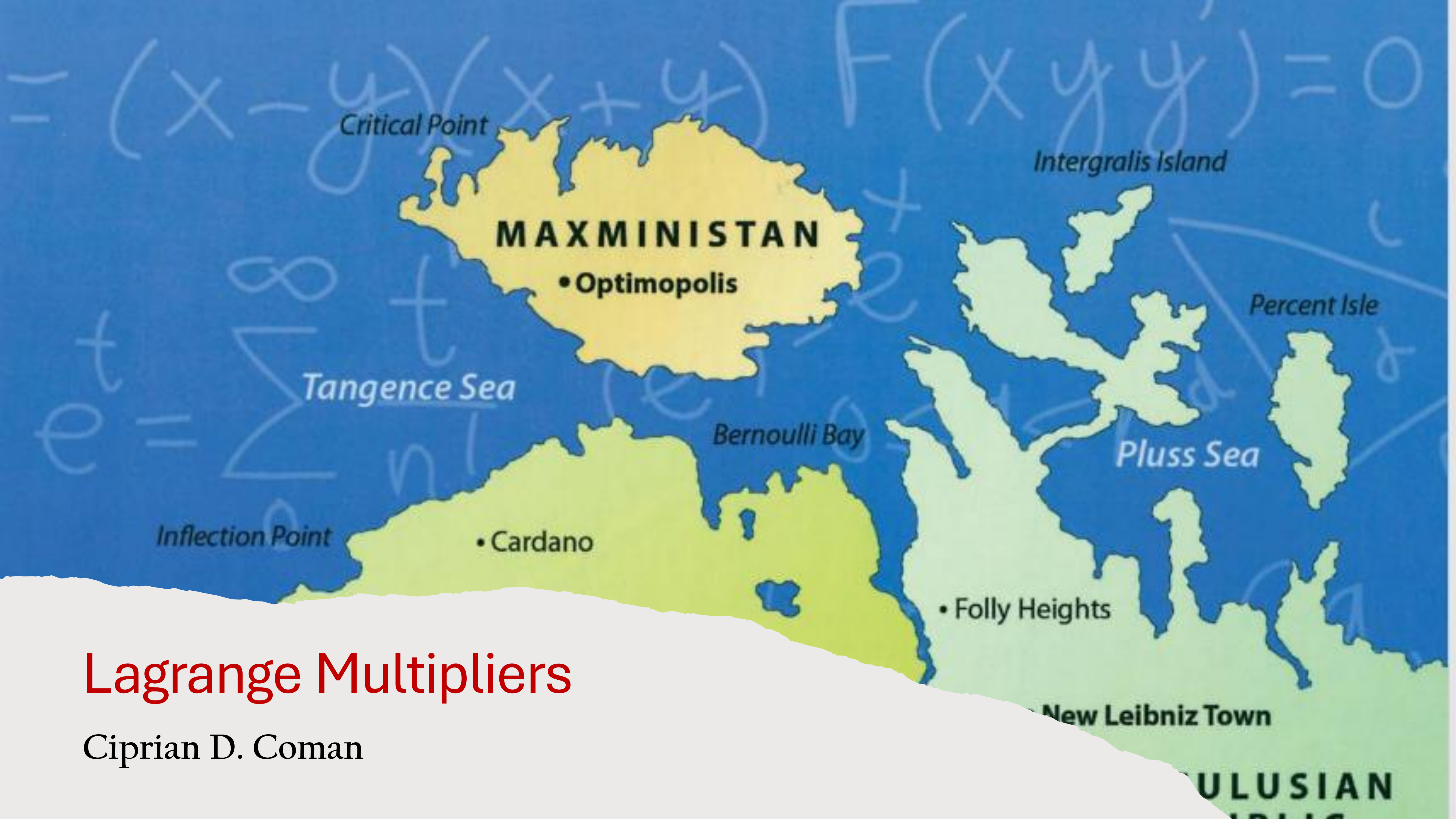


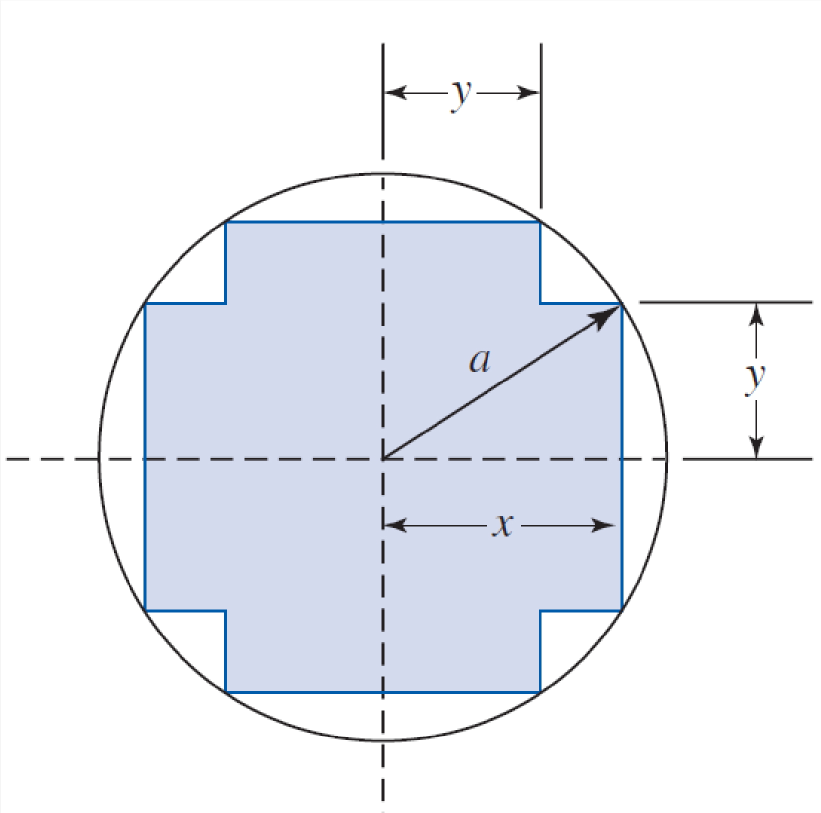
Lagrange Multipliers

Ciprian D. Coman

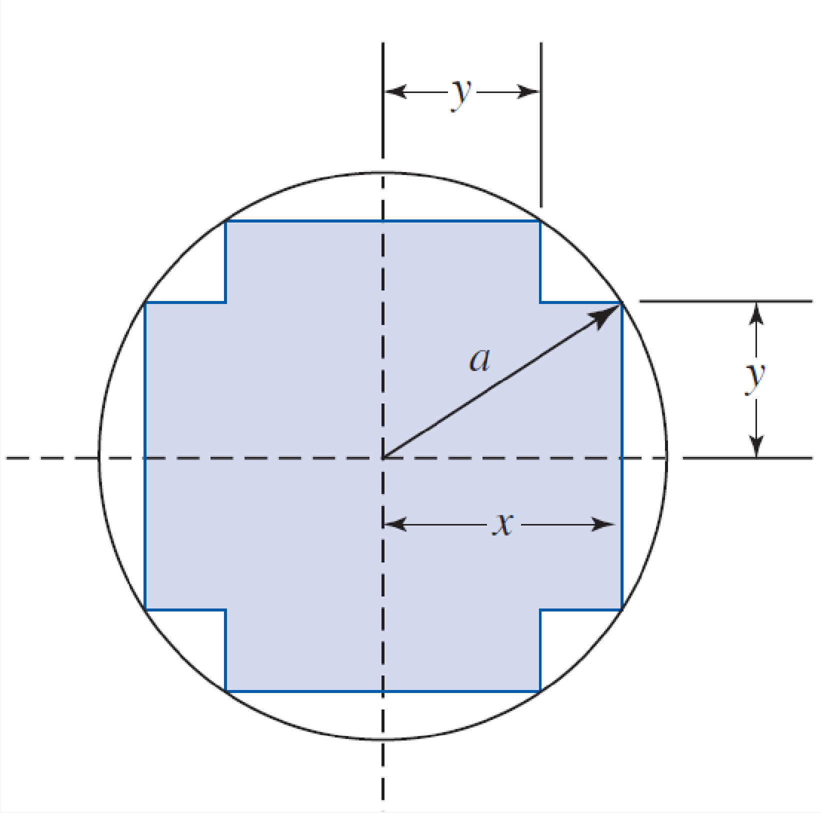
Constrained maxima & minima

An example (from electrical engineering):

We are required to find the cross-shaped iron core of *largest* surface area that can be inserted into a coil of radius a



Constrained maxima & minima



An example (from electrical engineering):

We are required to find the cross-shaped iron core of *largest* surface area that can be inserted into a coil of radius a

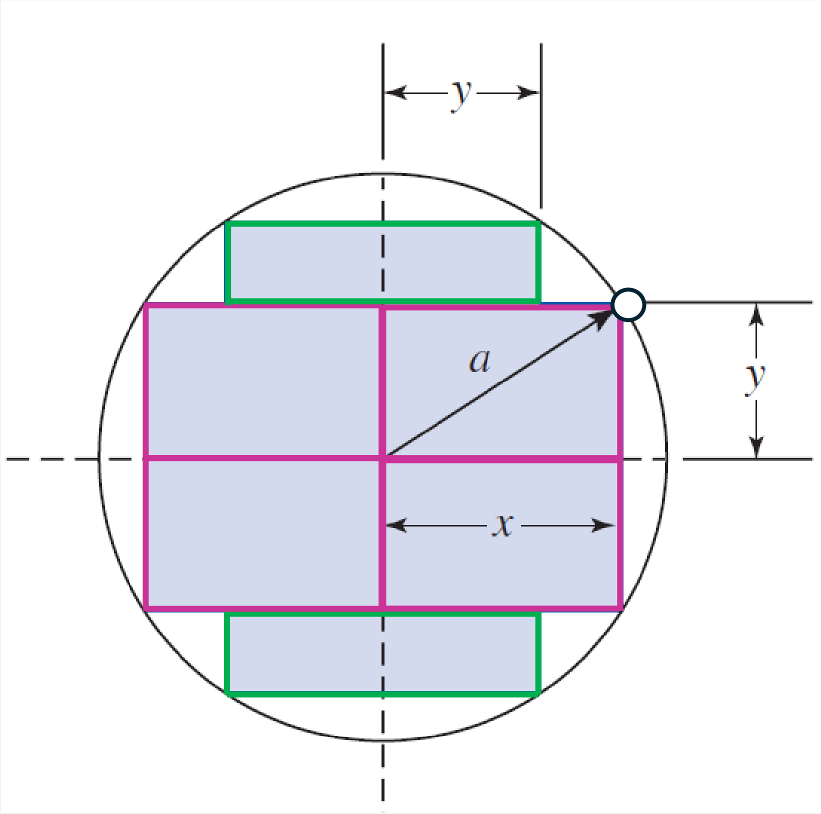
The problem is equivalent to maximizing the **objective function**

$$f(x, y) = 8xy - 4y^2$$

subject to the **constraint**

$$g(x, y) := x^2 + y^2 = a^2$$

Constrained maxima & minima



An example (from electrical engineering):

We are required to find the cross-shaped iron core of *largest* surface area that can be inserted into a coil of radius a

$$S = 4xy + 4y(x - y) = 8xy - 4y^2$$

Note that the point (x, y) is on a circle of radius a centred at the origin:

$$x^2 + y^2 = a^2$$

A typical situation
(constrained optimization)

Let f and g be given functions that have continuous first partial derivatives in some region $D \subset \mathbb{R}^2$. Find

$$\max \ f(x,y) \qquad \text{or} \qquad \min f(x,y)$$

subject to

$$g(x,y) = k \qquad (k \in \mathbb{R}, \text{given})$$

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Terminology:

$f(x,y)$ → the objective function

$g(x,y)$ → the constraint

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Observation: We can have *several* independent variables + *several* constraints

$$\max f(\boldsymbol{x}) \qquad \text{or} \qquad \min f(\boldsymbol{x}) \qquad \boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$$

subject to $g_j(\boldsymbol{x}) = k_j \quad \text{for } j = 1,2,\ldots,m \qquad (m < n)$

k_j → given constants

The need for a new approach...

To find the max/min of $f(x, y)$
we typically must identify its **critical points**

$$\nabla f(x, y) = \mathbf{0}$$

$$\big(f_x(x, y), f_y(x, y)\big) = (0, 0)$$

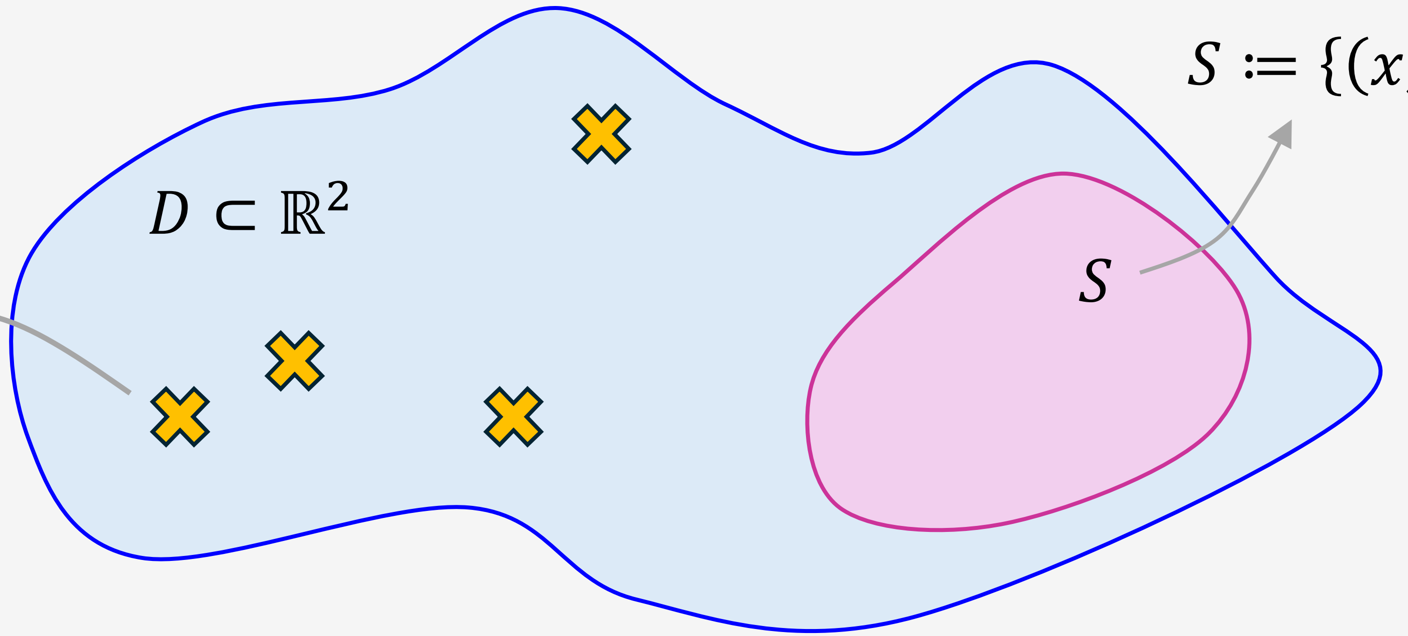
the **constraint set**

$$S := \{(x, y) \in D : g(x, y) = k\}$$

(subset of D)

critical points

(not in S!)



The main result (Lagrange's Theorem)

Let f and g be given functions that have continuous first partial derivatives in some region $D \subset \mathbb{R}^2$. If f has an **extremum** at a point $(x_0, y_0) \in S$ and $\nabla g(x_0, y_0) \neq \mathbf{0}$, then there is a $\lambda \in \mathbb{R}$ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

REMARKS: The number λ is called a **Lagrange multiplier**.

The **constraint set** was introduced on the previous slide:

$$S := \{(x, y) \in D : g(x, y) = k\}$$

The main result (Lagrange's Theorem)

Let f and g be given functions that have continuous first partial derivatives in some region $D \subset \mathbb{R}^2$. If f has an **extremum** at a point $(x_0, y_0) \in S$ and $\nabla g(x_0, y_0) \neq \mathbf{0}$, then there is a $\lambda \in \mathbb{R}$ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

$$\begin{cases} f_x(x_0, y_0) = \lambda g_x(x_0, y_0) \\ f_y(x_0, y_0) = \lambda g_y(x_0, y_0) \\ g(x_0, y_0) = k \end{cases}$$

(this is the original **constraint**)

→ 3 EQUATIONS IN 3 UNKNOWNS: x_0, y_0, λ

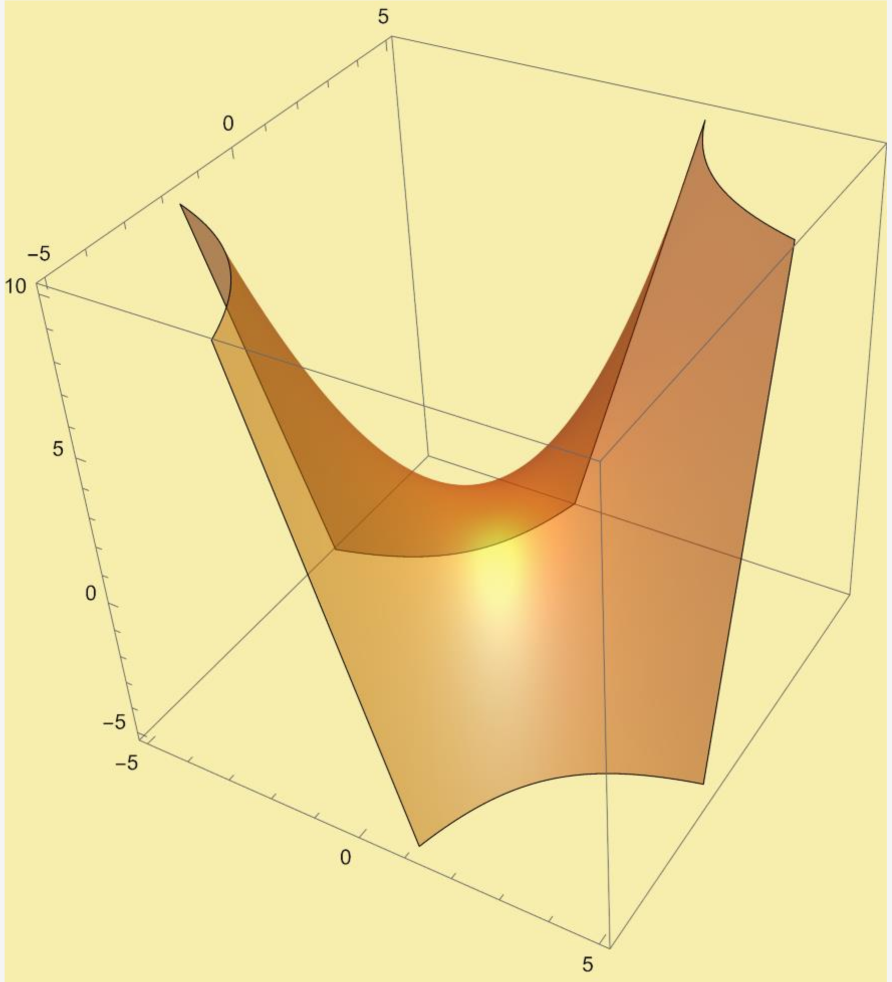
Example:

Find the **extrema** of $f(x, y) = xy + 1$, **subject to** $x^2 + y^2 = 4$

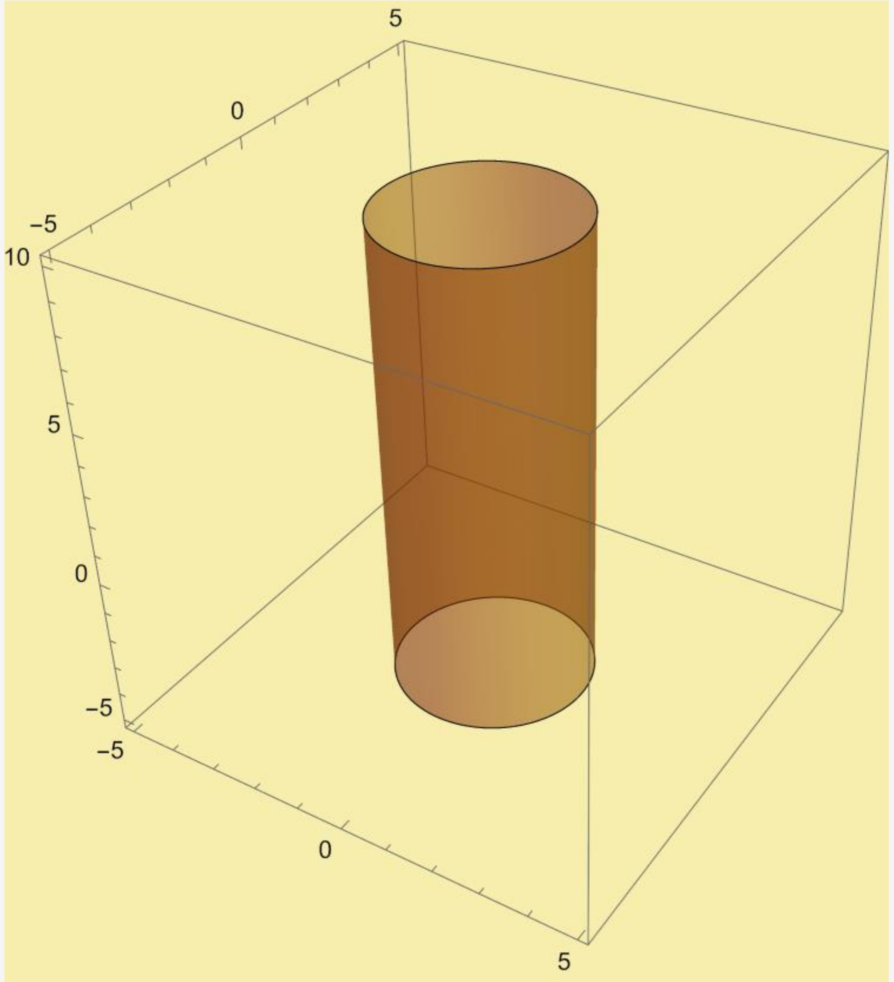
Note: Here, $g(x, y) = x^2 + y^2$ and $k = 4$

Geometrical interpretation

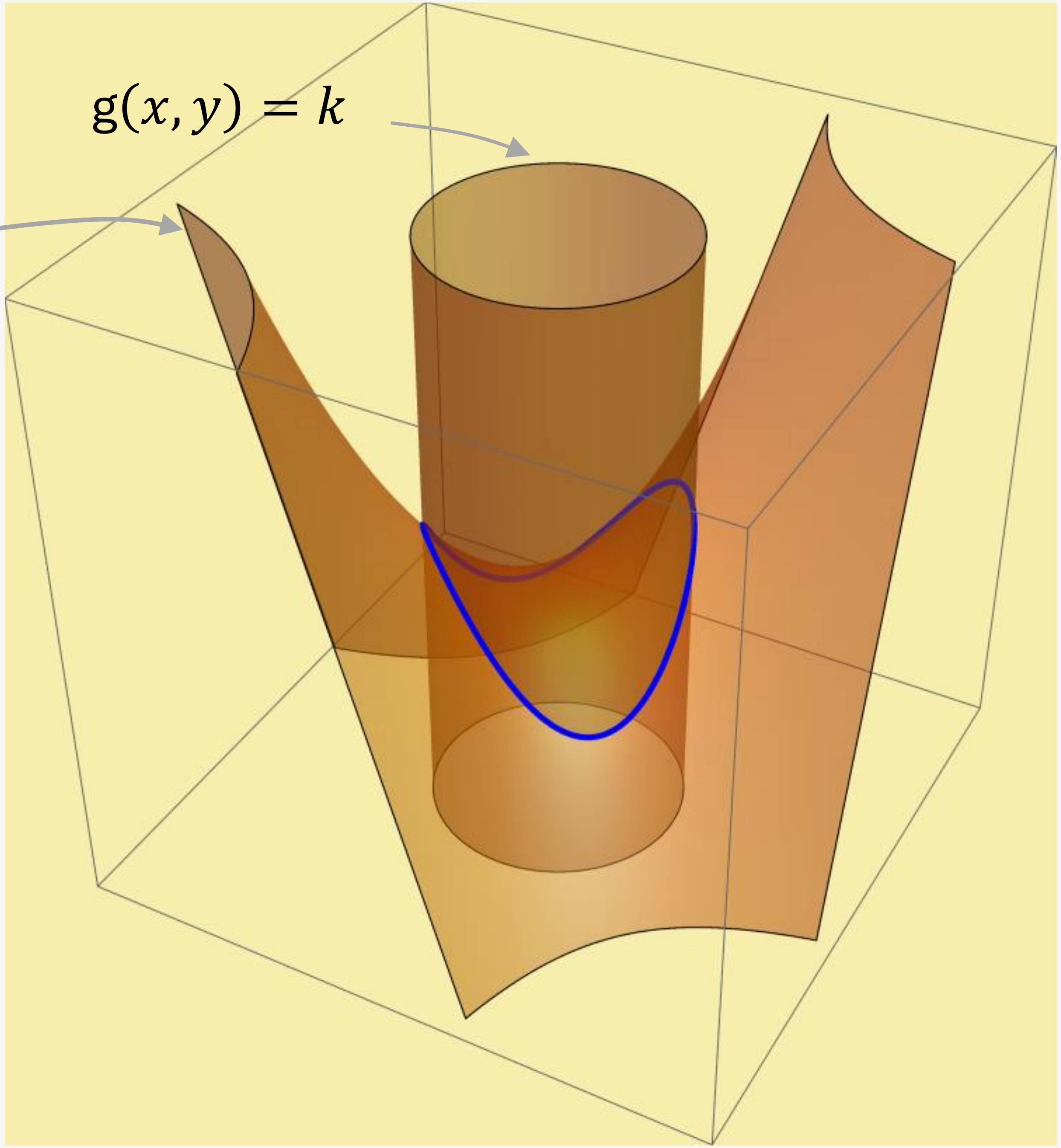
$$z = xy + 1$$



$$x^2 + y^2 = 4, \qquad z \in \mathbb{R}$$



Geometrical interpretation



The set of points $(x, y, z) \in \mathbb{R}^3$ that simultaneously satisfy:

$z = f(x, y)$ **AND** $g(x, y) = k$

⇩

blue curve

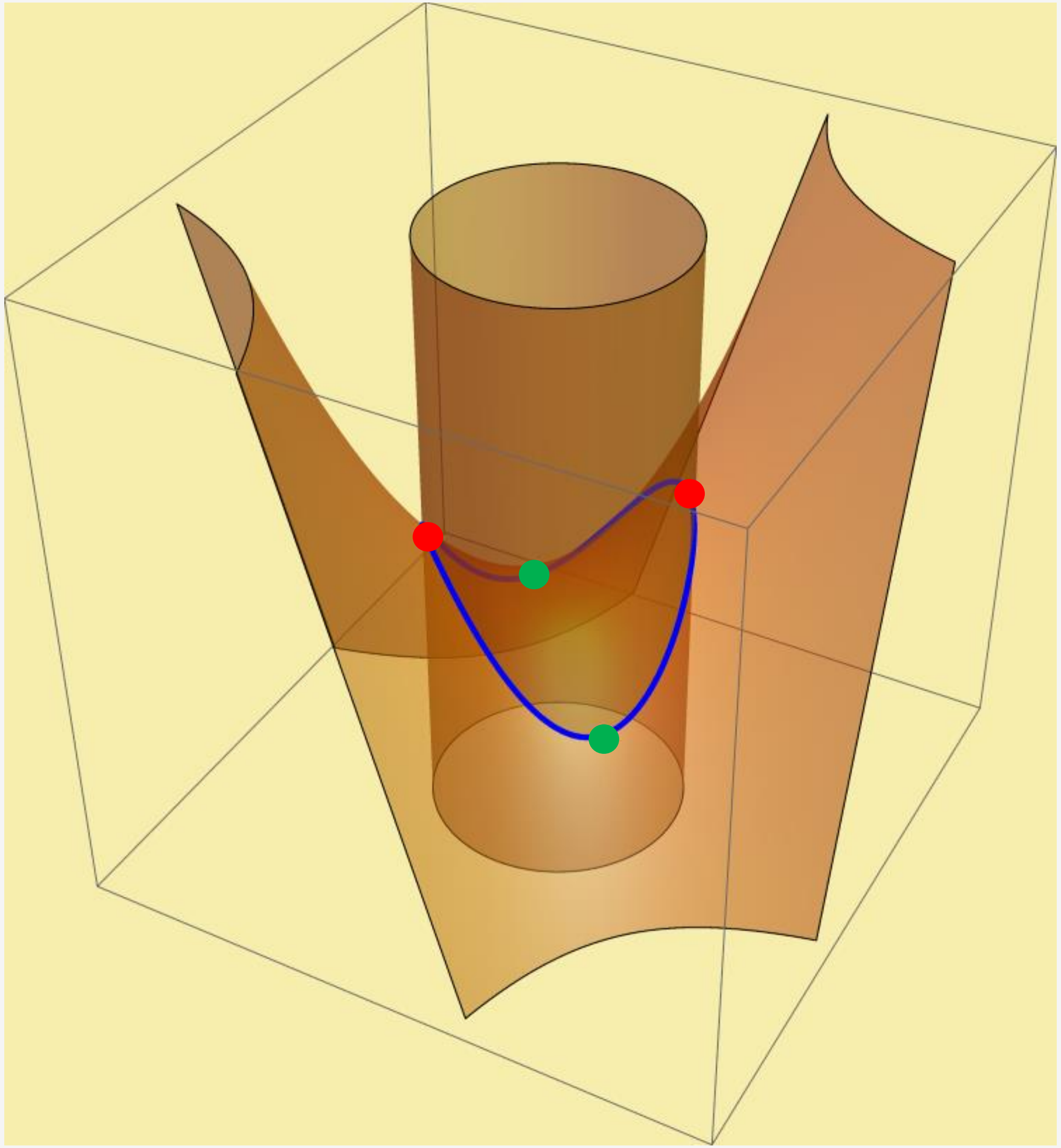
Re-interpretation

$$\max f(x,y) \quad \text{s.t.} \quad g(x,y) = k$$

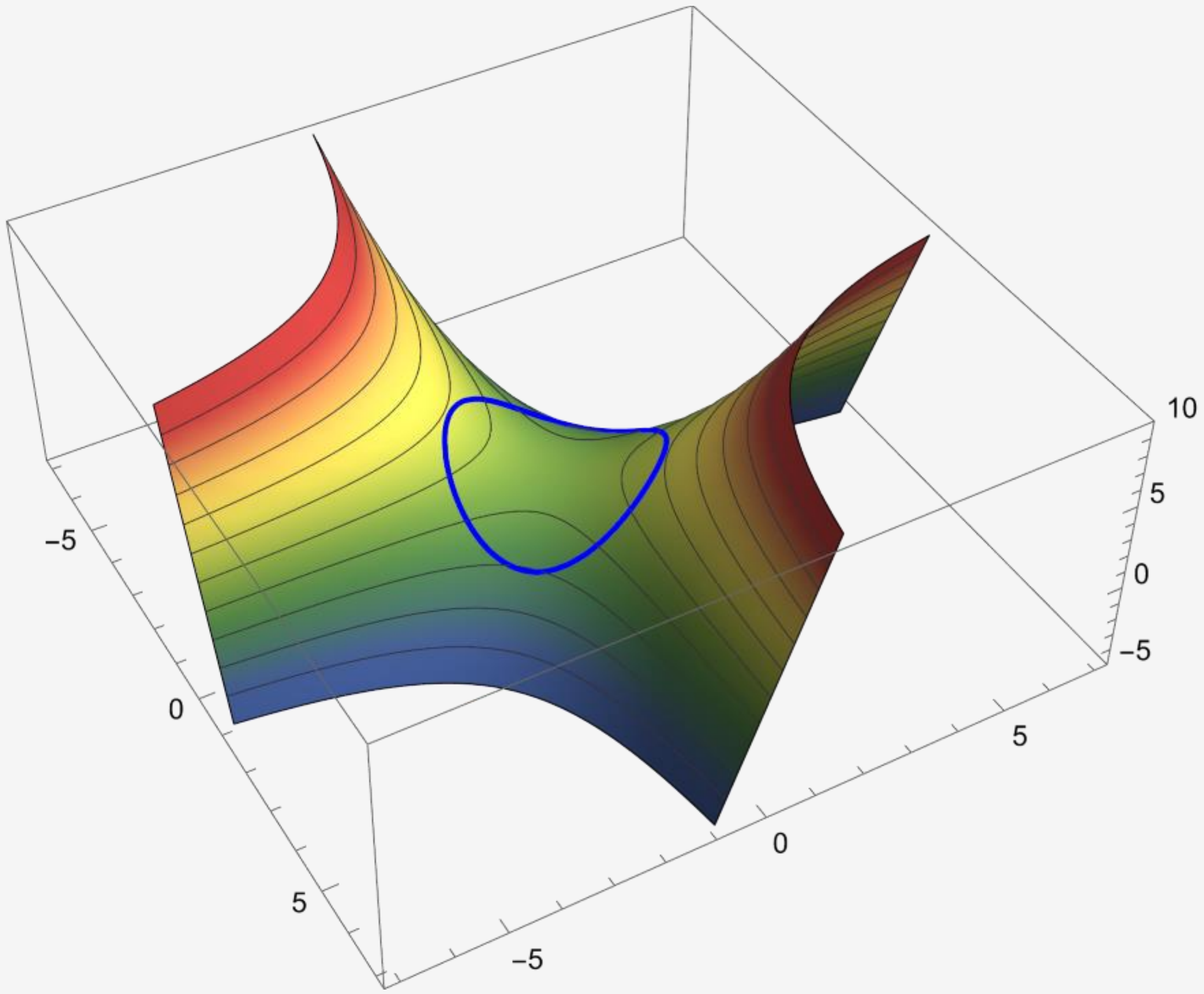
correspond to the **highest** points on the blue curve (RED dots)

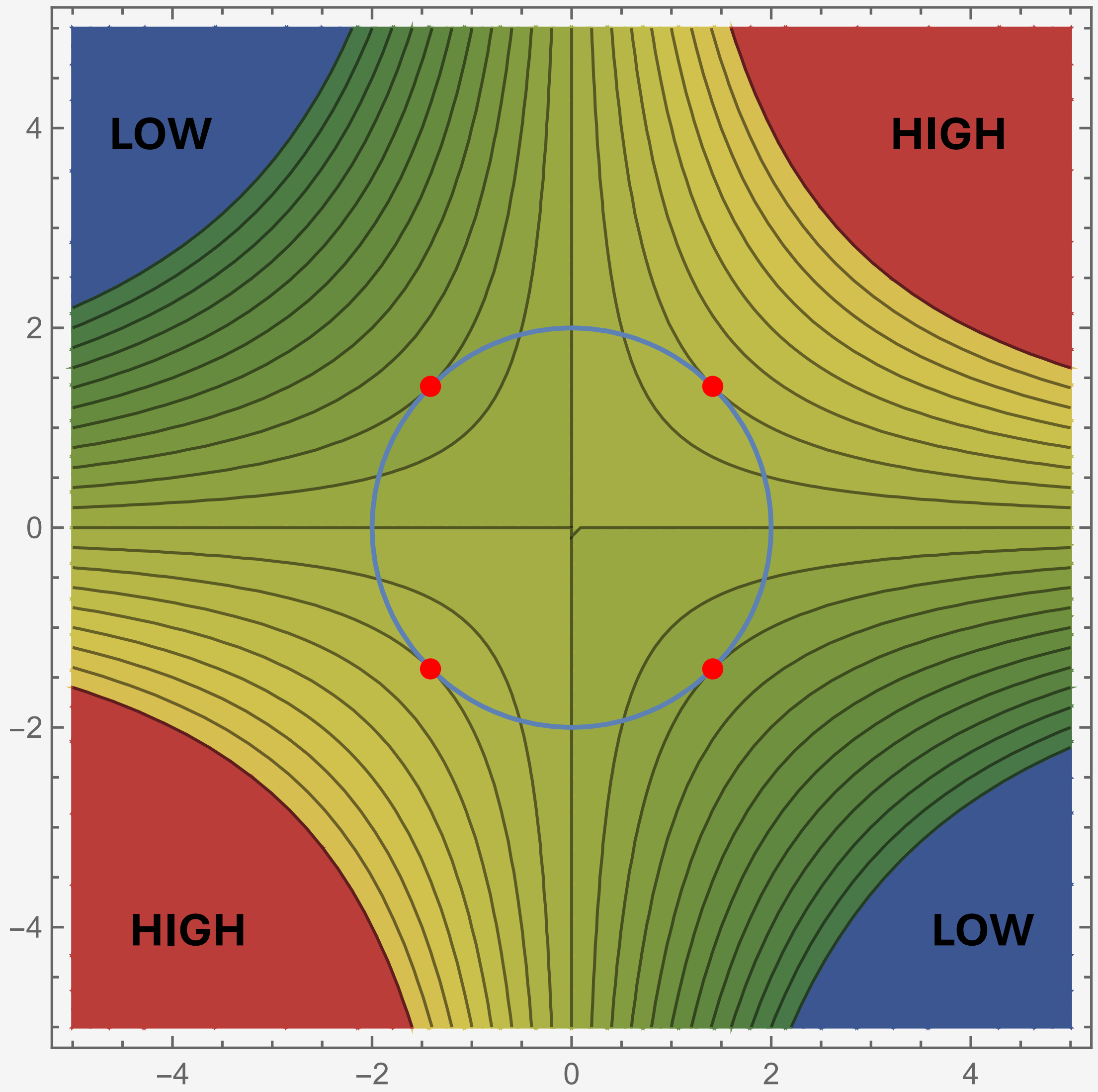
$$\min f(x,y) \quad \text{s.t.} \quad g(x,y) = k$$

correspond to the **lowest** points on the blue curve (GREEN dots)



Further geometrical considerations:





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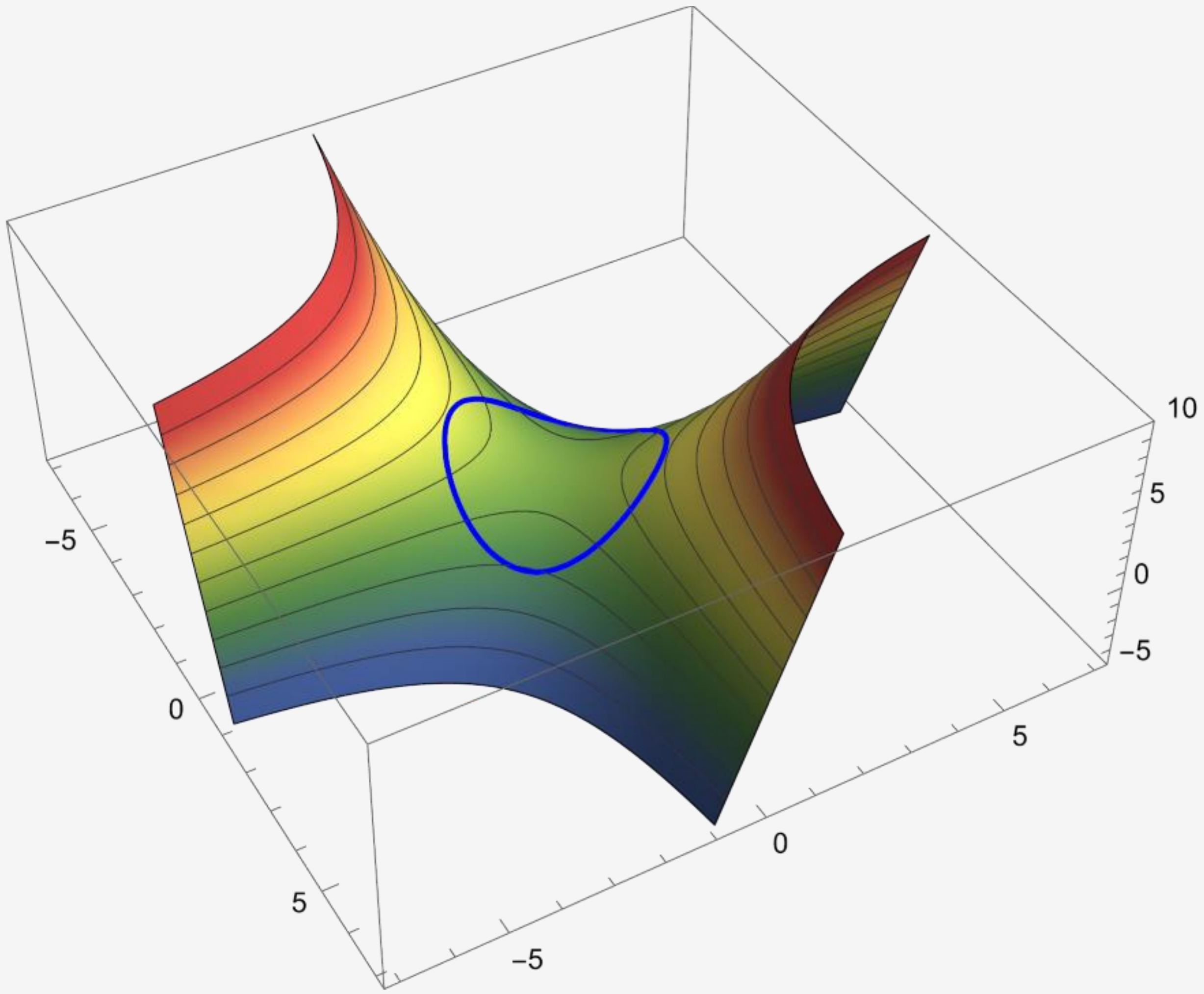

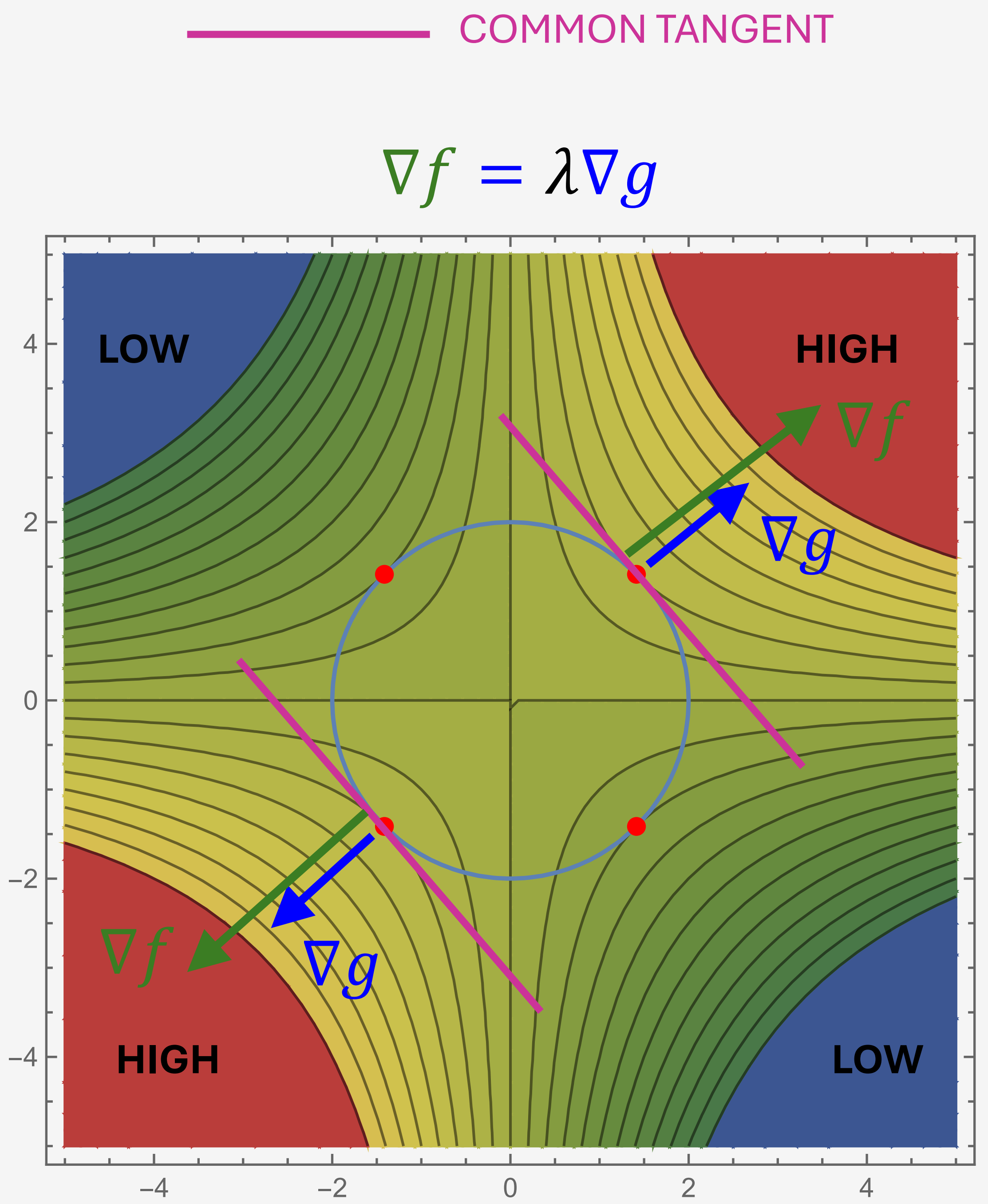

Example:

Find the **extrema** of $f(x,y) = xy + 1$, **subject to** $x^2 + y^2 = 4$

Solution: Here, $g(x,y) = x^2 + y^2$ and $k = 4$

$$\begin{cases} f_x(x_0,y_0) = \lambda g_x(x_0,y_0) \\ f_y(x_0,y_0) = \lambda g_y(x_0,y_0) \\ g(x_0,y_0) = k \end{cases} \Rightarrow \begin{cases} y_0 = \lambda(2x_0) \\ x_0 = \lambda(2y_0) \\ (x_0)^2+(y_0)^2 = 4 \end{cases} \Rightarrow \begin{cases} x_0 = 2\lambda y_0 \\ y_0 = 2\lambda x_0 \\ (x_0)^2+(y_0)^2 = 4 \end{cases}$$

Substitute the first two eqns. into the last one:

$$4\lambda^2((x_0)^2 + (y_0)^2) = 4 \Rightarrow 4\lambda^2 = 1 \Rightarrow \boxed{\lambda = \pm\frac{1}{2}}$$

Example:

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$$\begin{cases} x_0 = 2\lambda y_0 \\ y_0 = 2\lambda x_0 \\ (x_0)^2+(y_0)^2 = 4 \end{cases}$$

$$\lambda = \pm\frac{1}{2}$$

$\lambda = 1/2$ → $x_0 = y_0$

$(\sqrt{2}, \sqrt{2})$; $(-\sqrt{2}, -\sqrt{2})$

$\lambda = -1/2$ → $x_0 = -y_0$

$(-\sqrt{2}, \sqrt{2})$; $(\sqrt{2}, -\sqrt{2})$

Comparing the values of f at the points found above:

$f(\sqrt{2}, \sqrt{2}) = f(-\sqrt{2}, -\sqrt{2}) = 3$ **MAX**

$f(\sqrt{2}, \sqrt{2}) = f(-\sqrt{2}, -\sqrt{2}) = -1$ **MIN**