Introduction to Green's functions

An intuitive approach

Ciprian D Coman

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University of Huddersfield

Overview:

- Motivation
- Historical facts
- Physical interpretation
- Properties of the Green's functions
- A formal definition for second-order differential operators
- An algorithm for computing some Green's functions
- Worked example

Learning Objective:

To introduce the **concept of Green's function** in a simplified setting, together with its formal mathematical definition

1.2 Motivation

Linear algebra:

$\boldsymbol{A} \in M_{n\times n}(\mathbb{R})$, $\mathbf{b} \in M_{n\times 1}(\mathbb{R})$

$\boldsymbol{A}\boldsymbol{x} = \mathbf{b} \quad \Longrightarrow \quad \boxed{\boldsymbol{x} = \boldsymbol{A}^{-1}\mathbf{b}}$ provided that $\det \boldsymbol{A} \neq 0$ $\qquad (\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I})$

Physics/Engineering/Chemistry:

Boundary-Value Problems (BVPs), e.g.

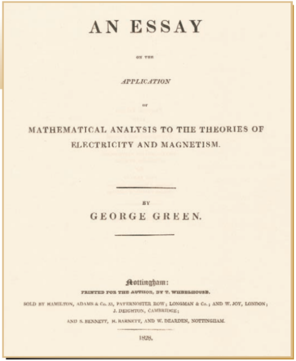
$$(*)\begin{cases} x\dfrac{d^2u}{dx^2} + \dfrac{du}{dx} + 5u = f(x)\,, & 1 < x < 3\,, \\ u(1) = 3, \quad u(3) = 1 \end{cases}$$

The ODE in $(*)$ can be written as

$$\boxed{\mathcal{L}u = f}\,, \qquad \text{where} \quad \mathcal{L} := x\frac{d^2}{dx^2} + \frac{d}{dx} + 5$$

Our aim: Would like to write the solution of $(*)$ as $u = \mathcal{L}^{-1}f$

1.3 History



AN ESSAY

ON THE

APPLICATION

OF

MATHEMATICAL ANALYSIS TO THE THEORIES OF ELECTRICITY AND MAGNETISM.

BY

GEORGE GREEN.

Nottingham:

PRINTED FOR THE AUTHOR, BY T. WHEELHOUSE.

SOLD BY HAMILTON, ADAMS & Co. 33, PATERNOSTER ROW; LONGMAN & Co.; AND W. JOY, LONDON; J. DEIGHTON, CAMBRIDGE;

AND S. BENNETT, H. BARNETT, AND W. DEARDEN, NOTTINGHAM.

1828.

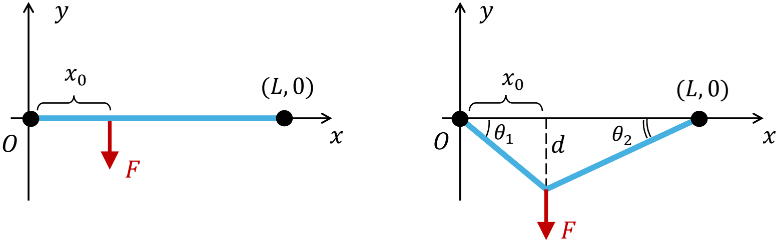
George Green:

- formal education: 18 months (before the age of 9)
- his major contribution was self-published at the age of 35
- attended Cambridge University when he was almost 40

1.4 Interpretation

Consider a taut metal string (initial tension $T > 0$) which is rigidly attached at $x = 0$, L.
A small point-force is applied on the string at $x = x_0$.

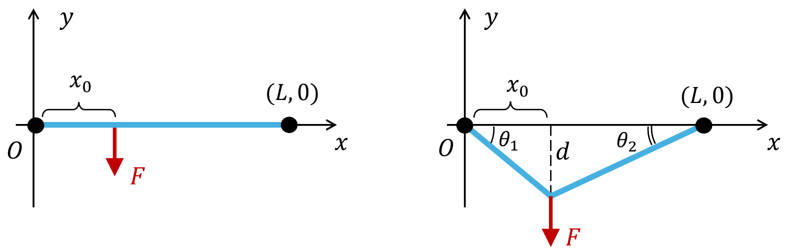
We want the **vertical displacement** of the string.



Assumption: the deflection is very small, so that every point on the string is constrained to move only in the vertical direction:

$$|\theta_1|, \; |\theta_2| \ll 1$$

1.4 Interpretation



$|\theta_1|, |\theta_2| \ll 1 \Longrightarrow \sin\theta \simeq \tan\theta$

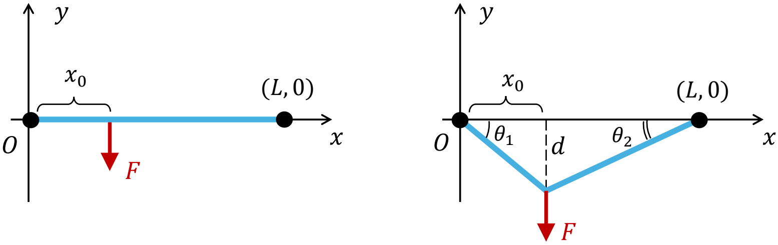
$$\sin\theta_1 = \frac{d}{x_0}\,, \qquad \sin\theta_2 = \frac{d}{L - x_0}$$

Equilibrium in the vertical direction:

$$T\sin\theta_1 + T\sin\theta_2 = F$$

Hence $d = \dfrac{F}{Ta}x_0(L - x_0)$.
We can now write down the equations of the two straight lines (see sketch on the right).

1.4 Interpretation



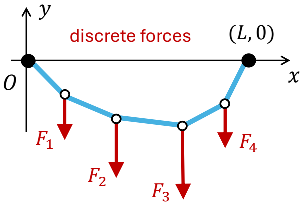
The deflection: $y(x) = F\,G(x, x_0)$, where

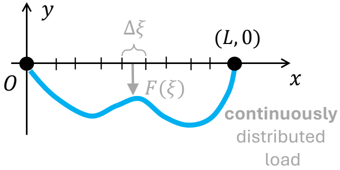
$$G(x, x_0) := \begin{cases} \dfrac{1}{TL} x(x_0 - L)\,, & 0 \le x < x_0\,, \\ \dfrac{1}{TL} x_0(x - L)\,, & x_0 < x \le L \end{cases}$$

Note

We can interpret $G(x, x_0)$ as the effect of a **unit force** acting at x_0 on the string.

1.4 Continuously distributed lateral load





Consider now the cumulative effect caused by several forces F_j acting at $x = x_j$ ($j = 1, 2, \ldots, n$). Assuming that the superposition principle works,

$$y(x) = \sum_{j=1}^{n} F_j G(x, x_j)$$

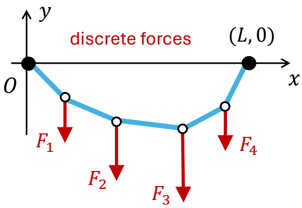
If the lateral load on the string is a continuously distributed function $F(x)$, then the total effect is given approximately by

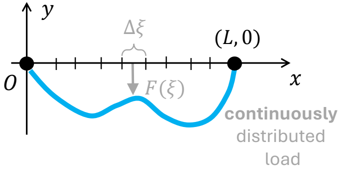
$$y(x) \simeq \sum \big(F(\xi)\Delta\xi\big) G(x, \xi)$$

Letting $\Delta\xi \to 0$ we obtain in the limit

$$\boxed{y(x) = \int_0^L F(\xi) G(x, \xi)\, d\xi}$$

1.5 Checking the integral solution





$$\boxed{y(x) = \int_0^L F(\xi) G(x,\xi)\, d\xi}$$

$$G(x,\xi) := \begin{cases} \dfrac{1}{TL} x(\xi - L)\,, & 0 \le x < \xi\,, \\ \dfrac{1}{TL} \xi(x - L)\,, & \xi < x \le L \end{cases}$$

The BVP for a taut string with a lateral load $F(x)$:

$$T\frac{d^2y}{dx^2} = F(x)\,, \qquad y(0) = y(L) = 0 \qquad (F \in C^0([0, L]))$$

Question: Does the solution in the box satisfy this equation?

1.5 Checking the integral solution

Split up the integral:

$$\boxed{y(x) = \int_0^x F(\xi)G(x,\xi)\,\mathrm{d}\xi + \int_x^L F(\xi)G(x,\xi)\,\mathrm{d}\xi}$$

Differrentiate w.r.t. x using the **Leibniz integral rule**

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(\int_0^x F(\xi)G(x,\xi)\,\mathrm{d}\xi\right) + \frac{d}{dx}\left(\int_x^L F(\xi)G(x,\xi)\,\mathrm{d}\xi\right) \\
&= \left[\int_0^x F(\xi)G_x(x,\xi)\,\mathrm{d}\xi + F(x)G(x,x^-)\right] + \left[\int_x^L F(\xi)G_x(x,\xi)\,\mathrm{d}\xi - F(x)G(x,x^+)\right] \\
&= \int_0^x F(\xi)G_x(x,\xi)\,\mathrm{d}\xi + \int_x^L F(\xi)G_x(x,\xi)\,\mathrm{d}\xi + F(x)\underbrace{\lim_{\varepsilon\to 0^+}\left[G(x,x-\varepsilon) - G(x,x+\varepsilon)\right]}_{G(x,\xi)\text{ continuous at }\xi=x} \\
&= \int_0^x F(\xi)G_x(x,\xi)\,\mathrm{d}\xi + \int_x^L F(\xi)G_x(x,\xi)\,\mathrm{d}\xi
\end{aligned}$$

1.5 Checking the integral solution

From the previous page:

$$\boxed{\frac{dy}{dx} = \int_0^x F(\xi)G_x(x,\xi)\,\mathrm{d}\xi + \int_x^L F(\xi)G_x(x,\xi)\,\mathrm{d}\xi}$$

Differrentiate w.r.t. x using the **Leibniz integral rule**

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\int_0^x F(\xi)G_x(x,\xi)\,\mathrm{d}\xi\right) + \frac{d}{dx}\left(\int_x^L F(\xi)G_x(x,\xi)\,\mathrm{d}\xi\right) \\
&= \left[\int_0^x F(\xi)G_{xx}(x,\xi)\,\mathrm{d}\xi + F(x)G_x(x,x^-)\right] + \left[\int_x^L F(\xi)G_{xx}(x,\xi)\,\mathrm{d}\xi - F(x)G_x(x,x^+)\right] \\
&= \int_0^x F(\xi)\underbrace{G_{xx}(x,\xi)}_{0}\,\mathrm{d}\xi + \int_x^L F(\xi)\underbrace{G_{xx}(x,\xi)}_{0}\,\mathrm{d}\xi + F(x)\underbrace{\lim_{\varepsilon\to 0^+}\left[G_x(x,x-\varepsilon) - G_x(x,x+\varepsilon)\right]}_{\text{is equal to } (1/T)} \\
&= \frac{1}{T}F(x) \quad \Longrightarrow \quad \boxed{T\frac{d^2y}{dx^2} = F(x)}
\end{aligned}$$

1.6 Context

Let $a, b \in \mathbb{R}$ and $p \in C^2([a,b])$, $q \in C^0([a,b])$, with $p(x) \neq 0$ $(\forall)\, x \in [a,b]$.

Differential operator:

$$\mathcal{L}_x := \frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x) \equiv p(x)\frac{d^2}{dx^2} + p'(x)\frac{d}{dx} + q(x)\,,$$

We are interested in solving

$$\boxed{\begin{array}{ll} \mathcal{L}_x[u(x)] = f(x)\,, & a < x < b \\ u(a) = u(b) = 0 & \end{array}}$$

for some given $f \in C^0([a,b])$. We'll call this problem **SL-1**.

The **general solution** of $\mathcal{L}_x[u] = f$ is of the form

$$\begin{aligned} u(x) &= u_{hom}(x) + u_{part}(x) \\ &= u_{hom}(x) + \int_a^b G(x,\xi) f(\xi)\, \mathrm{d}\xi \end{aligned}$$

1.6 Definition

The **Green's function** for the **SL-1 problem** is a function $G(x, \xi)$, $a \leq x, \xi \leq b$ which satisfies the following properties:

1. $G(x, \xi)$ is continuous for $a \leq x \leq b$; in particular
$$\lim_{x \to \xi^+} G(x, \xi) = \lim_{x \to \xi^-} G(x, \xi)$$
2. $\mathcal{L}_x G(x, \xi) = 0$ for all $x \neq \xi$
3. $G_x(x, \xi)$ is continuous except for a (finite) discontinuity at $x = \xi$; more precisely
$$\lim_{x \to \xi^+} G_x(x, \xi) - \lim_{x \to \xi^-} G_x(x, \xi) = \frac{1}{p(\xi)}$$

1.7 Algorithm for finding $G(x, \xi)$:

STEP 1: Find the two linearly independent solutions of the homogeneous equation $\mathcal{L}_x G(x, \xi) = 0$.

STEP 2: Take a linear combination of these solutions in order to find a solution that satisfies the **left** (at $x = a$) and **right** (at $x = b$) homogeneous boundary conditions. Call these $u_L(x)$ and $u_R(x)$, respectively.

STEP 3: Write the Green's function as

$$G(x, \xi) = \begin{cases} c_L(\xi) u_L(x) \,, & a \leq x \leq \xi \\ c_R(\xi) u_R(x) \,, & \xi \leq x \leq b \,, \end{cases}$$

where $c_L(\xi)$ and $c_R(\xi)$ are "constants" to be found (see below).

STEP 4: Enforce the continuity of $G(x, \xi)$ at $x = \xi$:

$$c_L(\xi) u_L(\xi) = c_R(\xi) u_R(\xi) \qquad (I)$$

STEP 5: Enforce the condition on G_x at $x = \xi$:

$$c_L(\xi) \frac{du_R}{dx}(\xi) - c_L(\xi) \frac{du_L}{dx}(\xi) = \frac{1}{p(\xi)} \qquad (II)$$

STEP 6: Solve for $c_L(\xi)$ and $c_R(\xi)$ by treating (I) and (II) as a systems of two linear equations in two unknowns (see solution on next page....)

1.7 Algorithm for finding $G(x,\xi)$:

(cont'd) The solution of the system at **STEP 6** can be shown to be

$$c_L(\xi) = \frac{u_R(\xi)}{p(\xi)W(\xi)}\,, \qquad c_R(\xi) = \frac{u_L(\xi)}{p(\xi)W(\xi)}$$

where we have defined the associated **Wronksian**

$$W(\xi) = \begin{vmatrix} u_L(\xi) & u_R(\xi) \\ u_L'(\xi) & u_R'(\xi) \end{vmatrix}$$

In conclusion, the Green's function for **SL-1** is given by

$$\boxed{G(x,\xi) = \begin{cases} \dfrac{u_R(\xi)u_L(x)}{p(\xi)W(\xi)}\,, & a \leq x \leq \xi \leq b \\[2ex] \dfrac{u_L(\xi)u_R(x)}{p(\xi)W(\xi)}\,, & a \leq \xi \leq x \leq b \end{cases}}$$

1.8 Example

Find the Green's function associated with the BVP

$$
\boxed{\begin{array}{l} x^2 \dfrac{d^2 u}{dx^2} + 2x \dfrac{du}{dx} - 2u = f(x)\,, \qquad 1 < x < 2 \\ u(1) = u(2) = 0 \end{array}}
$$

Solution: The above equation may be written as

$$
\frac{d}{dx}\left(x^2 \frac{du}{dx}\right) - 2u = f(x)
$$

(therefore we can appeal to the algorithm described)

Since the ODE is of **Euler type** we seek solutions in the form $u(x) = x^m$, for some m (to be determined next). We find that $m^2 + m - 2 = 0$ or $(m+2)(m-1) = 0$, and therefore $m = -2$ or $m = 1$; in the earlier notation, $u_1(x) = 1/x^2$ and $u_2(x) = x$. These can be combined to give the solutions

$$
u_L(x) = x - \frac{1}{x^2} \qquad \text{and} \qquad u_R(x) = x - \frac{8}{x^2}
$$

and we find that $p(\xi)W(\xi) = 21$.

The formulae from **STEP 6** yield

$$
c_L(\xi) = \frac{1}{21} u_R(\xi) = \frac{\xi^3 - 8}{21\xi^2}\,, \qquad c_R(\xi) = \frac{1}{21} u_L(\xi) = \frac{\xi^3 - 1}{21\xi^2}
$$

1.8 Example (cont'd)

The expression of the Green's function can be stated as

$$G(x,\xi) = \begin{cases} \dfrac{(x^3-1)(\xi^3-8)}{21x^2\xi^2}, & 1 \leq x \leq \xi \leq 2 \\[2ex] \dfrac{(\xi^3-1)(x^3-8)}{21x^2\xi^2}, & 1 \leq \xi \leq x \leq 2 \end{cases}$$

For $f(x) = x^2$ we find

$$\begin{aligned} u_{part}(x) &= \int_1^2 G(x,\xi)f(\xi)\,\mathrm{d}\xi \\ &= \int_1^x G(x,\xi)f(\xi)\,\mathrm{d}\xi + \int_x^2 G(x,\xi)f(\xi)\,\mathrm{d}\xi \\ &= \cdots = \frac{x^2}{4} \end{aligned}$$

1.9 Additional details...

$$\begin{aligned}\lim_{x\to\xi^-} G_x(x,\xi) &= \lim_{\substack{x\to\xi\\ x<\xi}} G_x(x,\xi)\\ &= \lim_{\substack{\xi\to x\\ \xi>x}} G_x(x,\xi)\\ &= G_x(x,x^+)\\ &= \lim_{\varepsilon\to 0^+} G_x(x,x+\varepsilon)\end{aligned}$$

$$\begin{aligned}\lim_{x\to\xi^+} G_x(x,\xi) &= \lim_{\substack{x\to\xi\\ x>\xi}} G_x(x,\xi)\\ &= \lim_{\substack{\xi\to x\\ \xi<x}} G_x(x,\xi)\\ &= G_x(x,x^-)\\ &= \lim_{\varepsilon\to 0^+} G_x(x,x-\varepsilon)\end{aligned}$$

$$G(x,\xi)=\begin{cases}\dfrac{1}{TL}x(\xi-L)\,, & 0\le x<\xi\,,\\[2mm] \dfrac{1}{TL}\xi(x-L)\,, & \xi<x\le L\end{cases} \quad\Longrightarrow\quad G_x(x,\xi)=\begin{cases}\dfrac{\xi-L}{TL}\,, & 0\le x<\xi\,,\\[2mm] \dfrac{\xi}{TL}\,, & \xi<x\le L\end{cases}$$

Hence

$$G_x(x,x^-)-G_x(x,x^+)=\frac{x}{TL}-\frac{x-L}{TL}=\frac{1}{T}$$