University of Huddersfield

The Fundamental Theorem of Calculus

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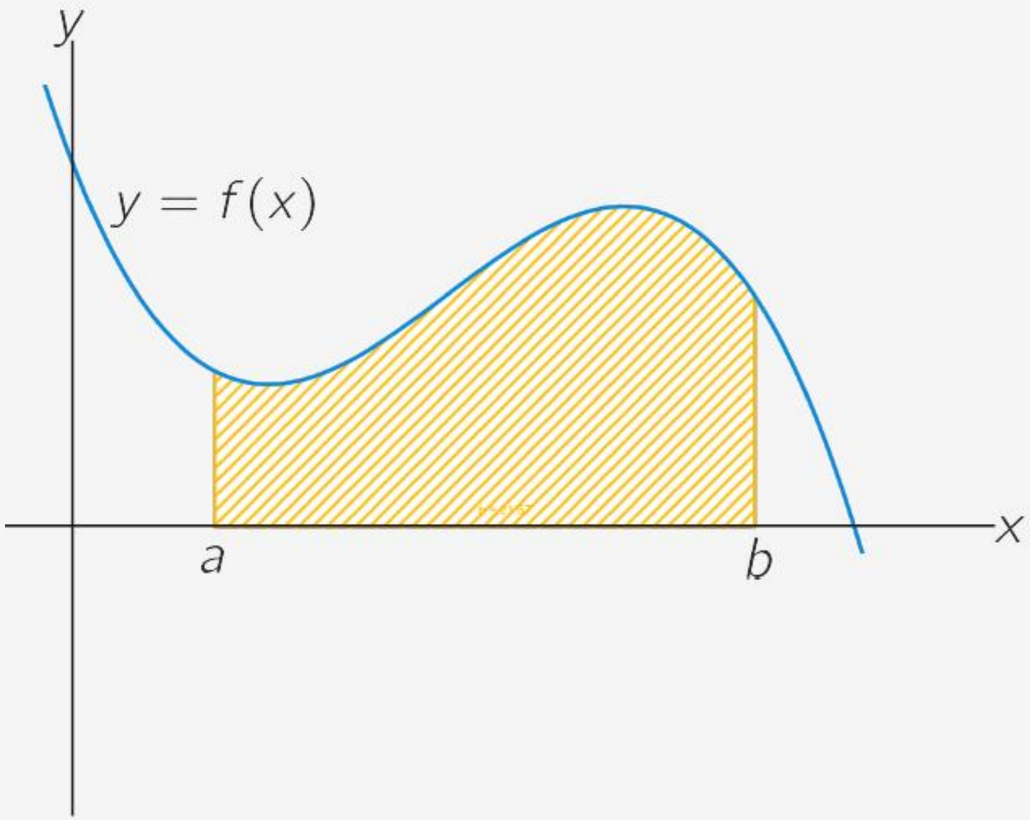
Overview

- Notation/terminology
- Statement of the **main result**
- Example
- Justification

The definite integral



$$\int_a^b f(x)\,dx$$

The definite integral

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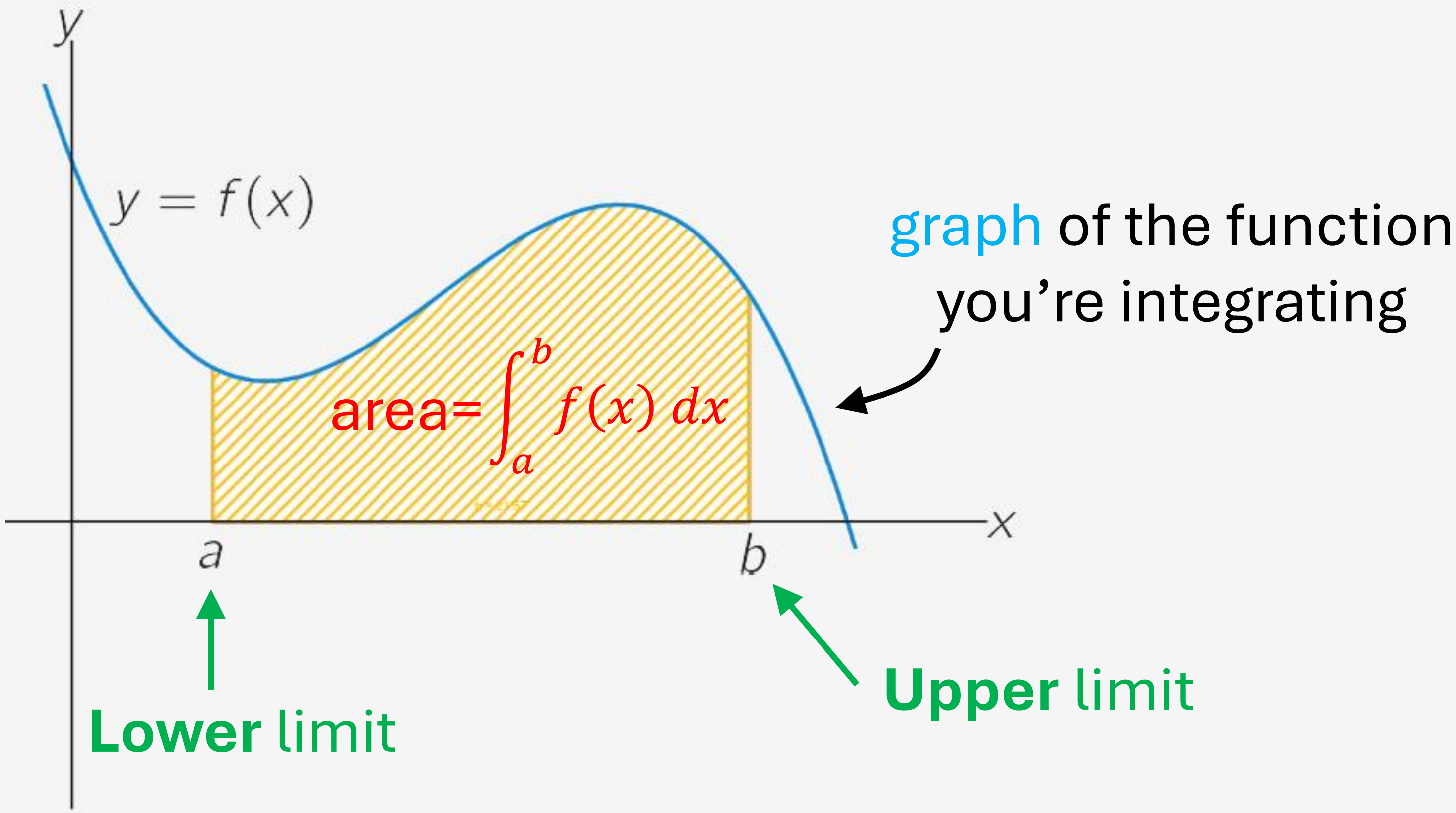

The Fundamental Theorem of Calculus (FTC)

- ❑ $f(x)$ continuous on $[a, b]$
- ❑ $F(x)$ function such that $F'(x) = f(x)$

then $$\int_a^b f(x)dx = F(b) - F(a)$$

primitive function
or **antiderivative** of $f(x)$

Example 1

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$(n \neq -1)$

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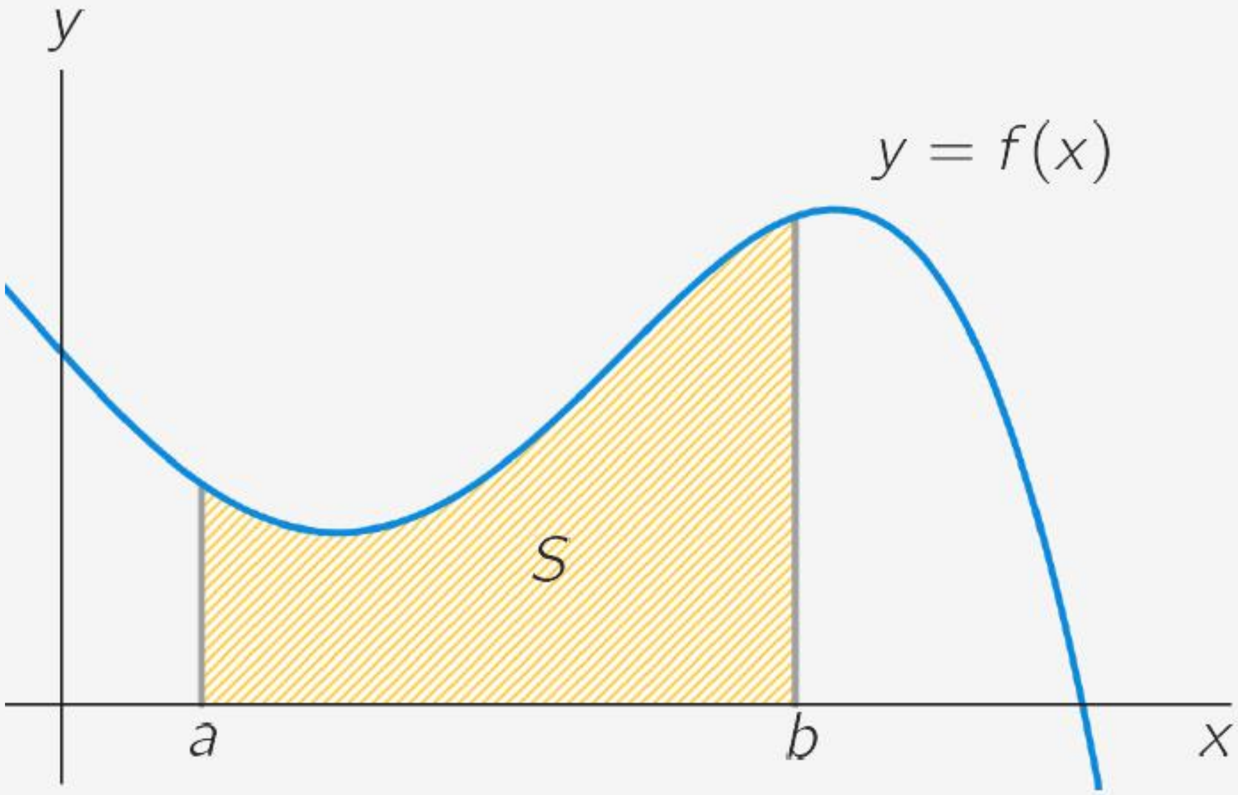
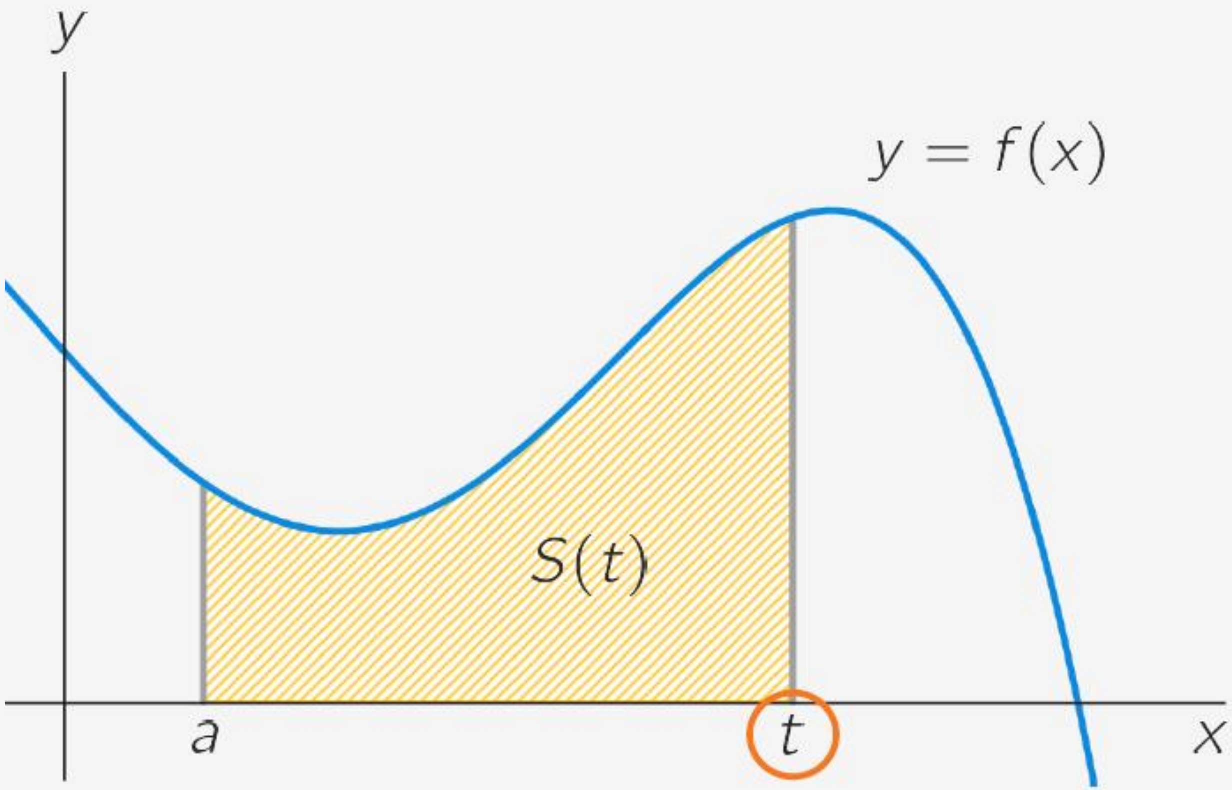
final answer

Justification for the FTC: why is it true?



$$S = \int_a^b f(x)dx$$

Justification for the FTC



$$S(t) = \int_{a}^{t} f(x)dx$$

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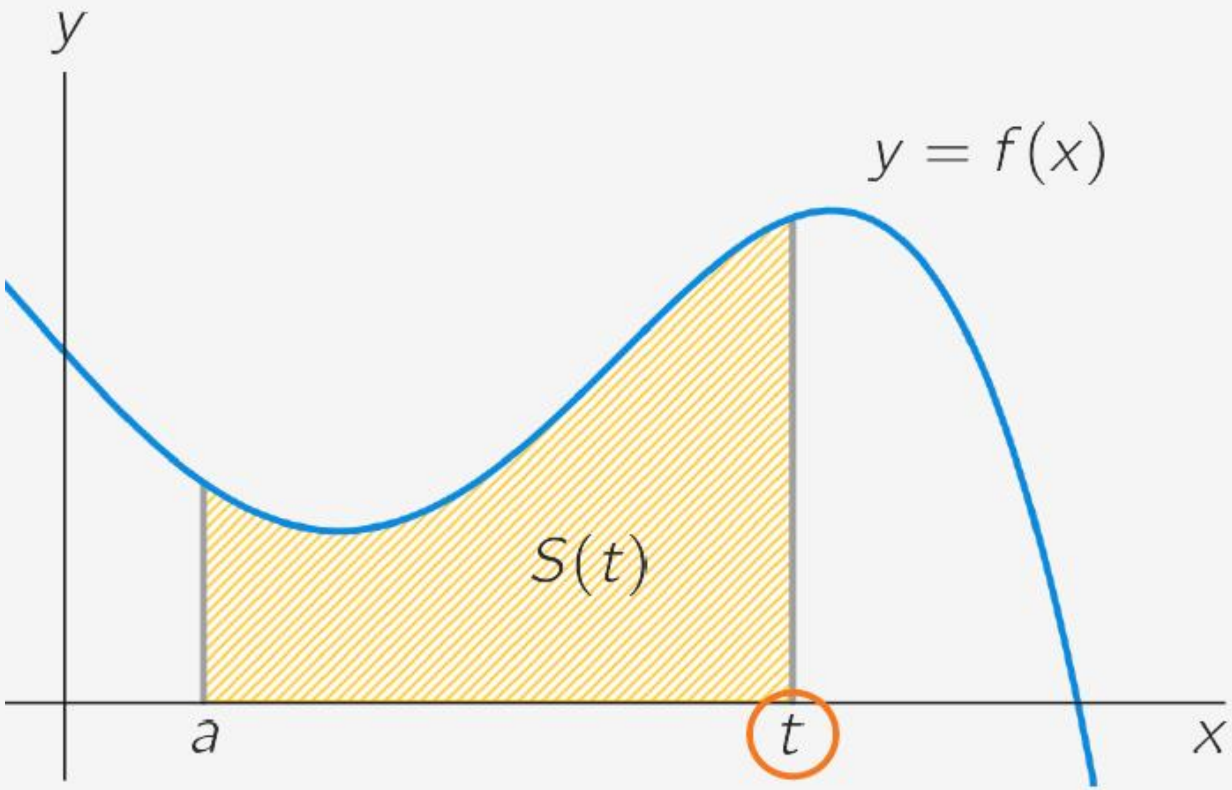

$$S(t) = \int_a^t f(x)dx$$

Claim: $S'(t) = f(t)$

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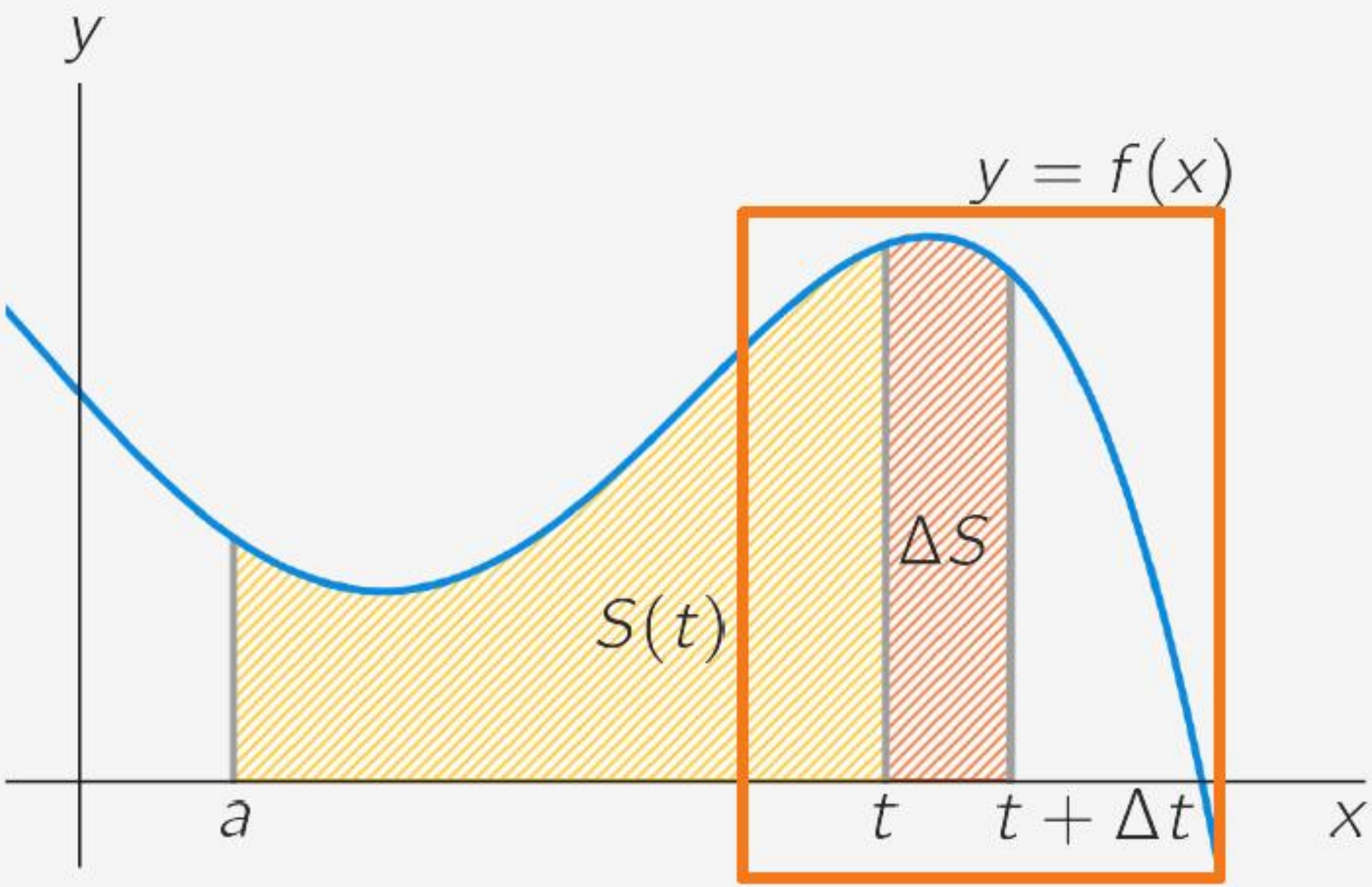

$\Delta S = S(t + \Delta t) - S(t)$

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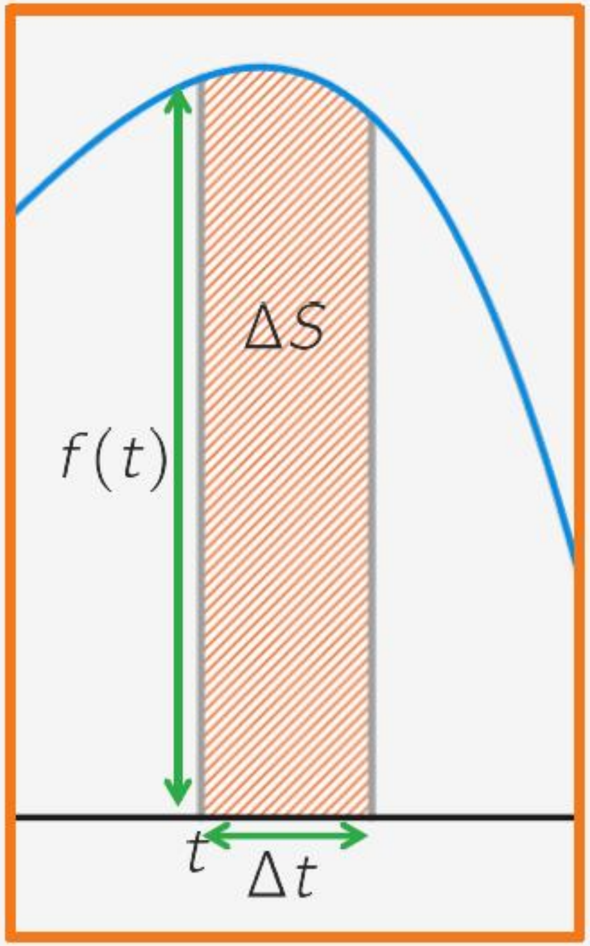

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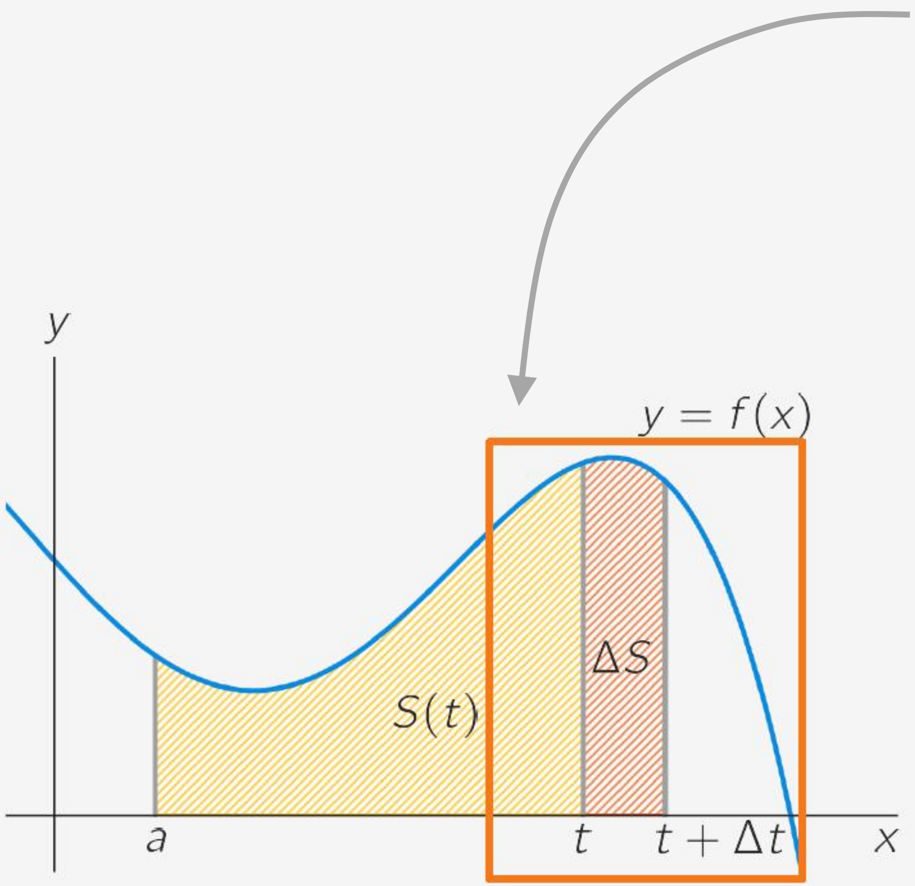

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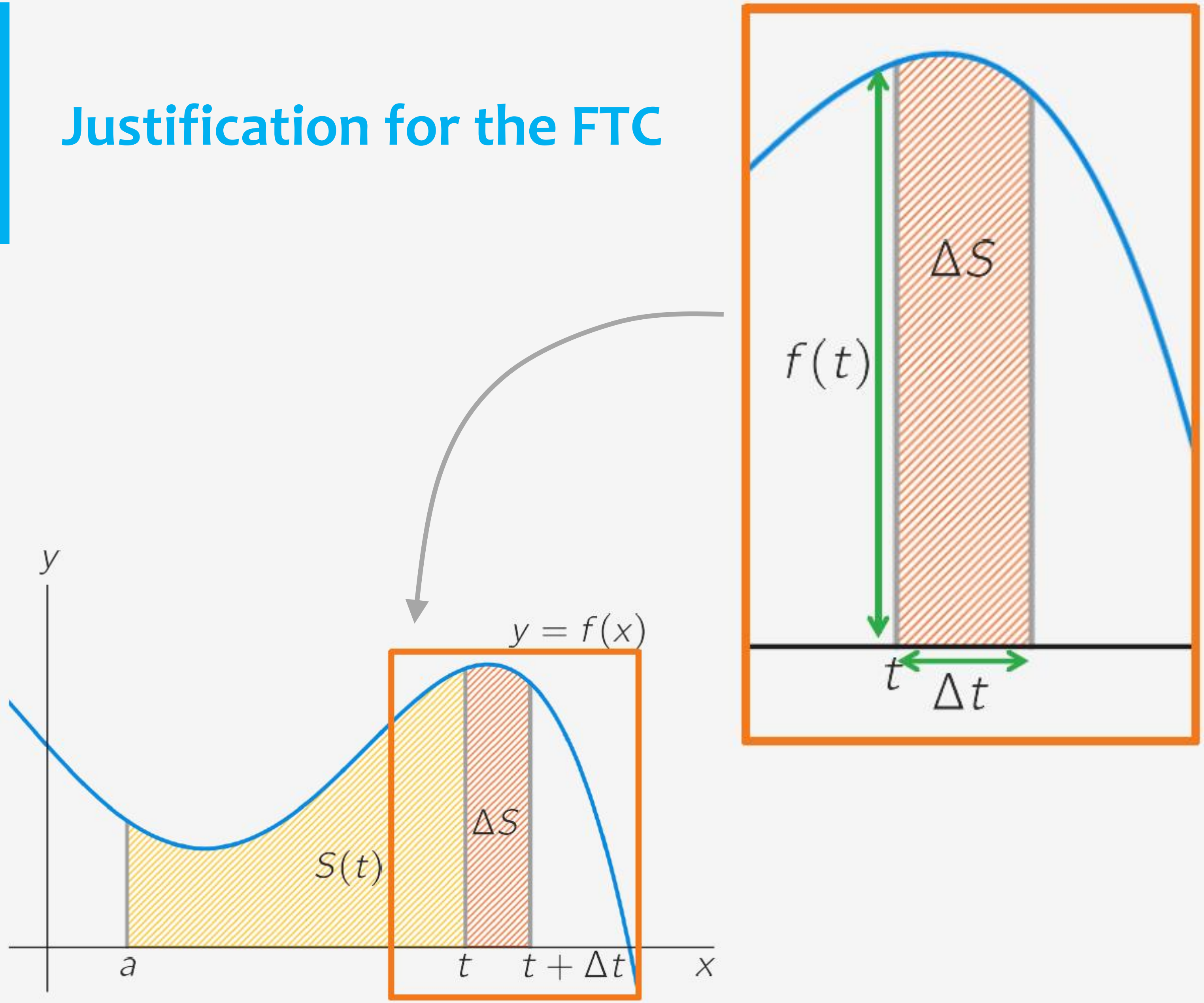

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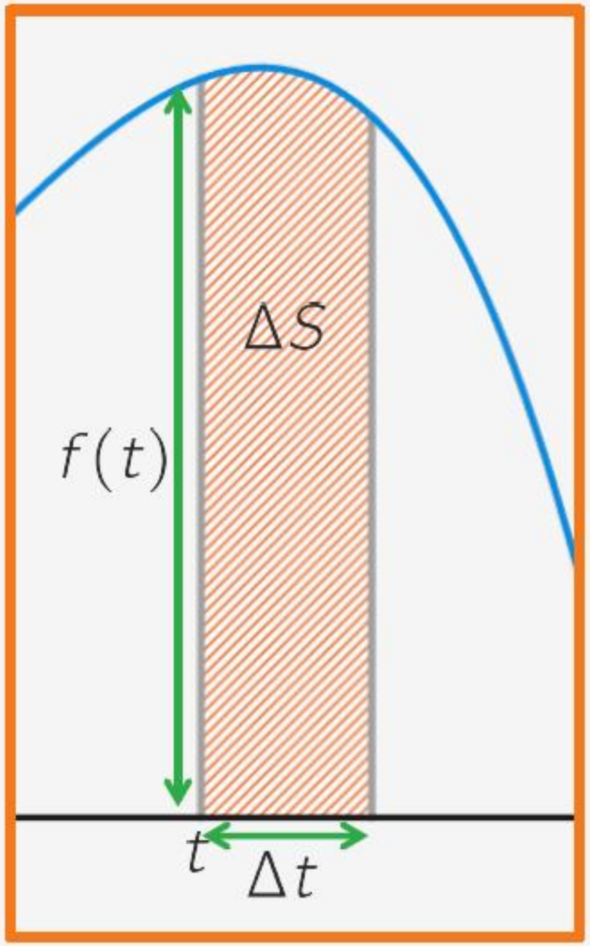

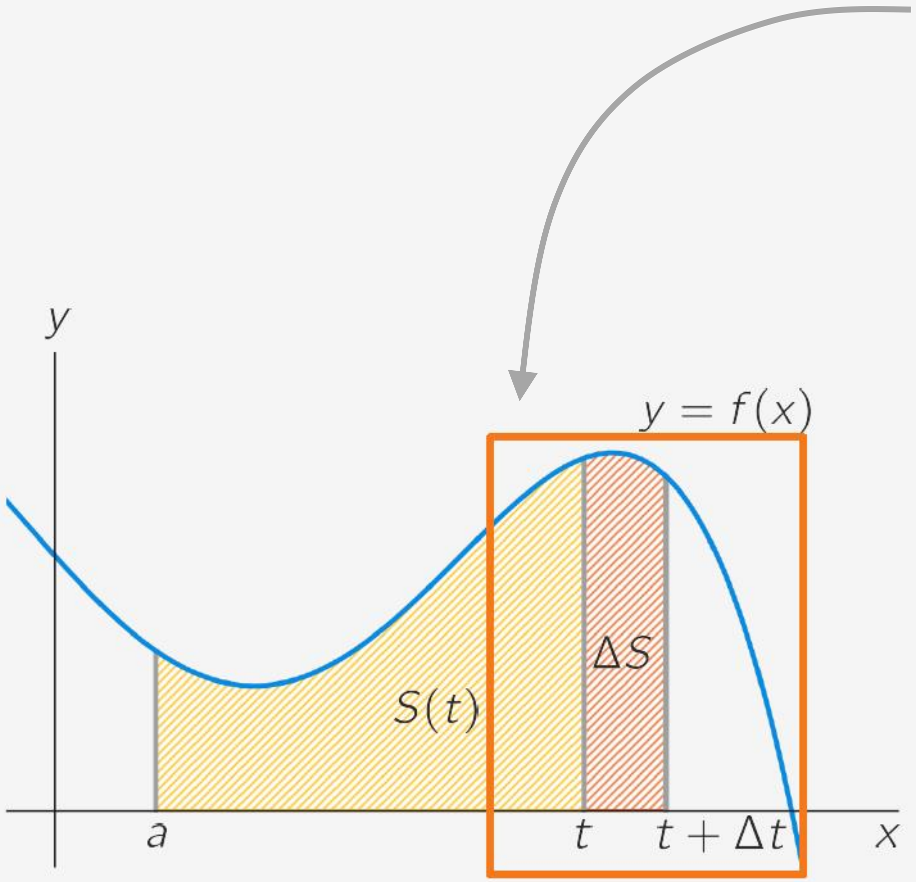

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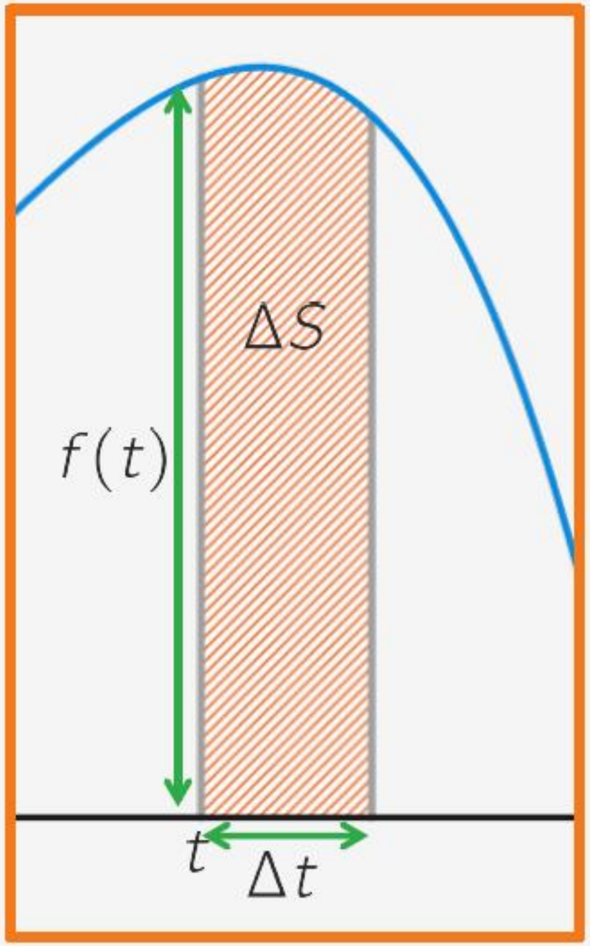

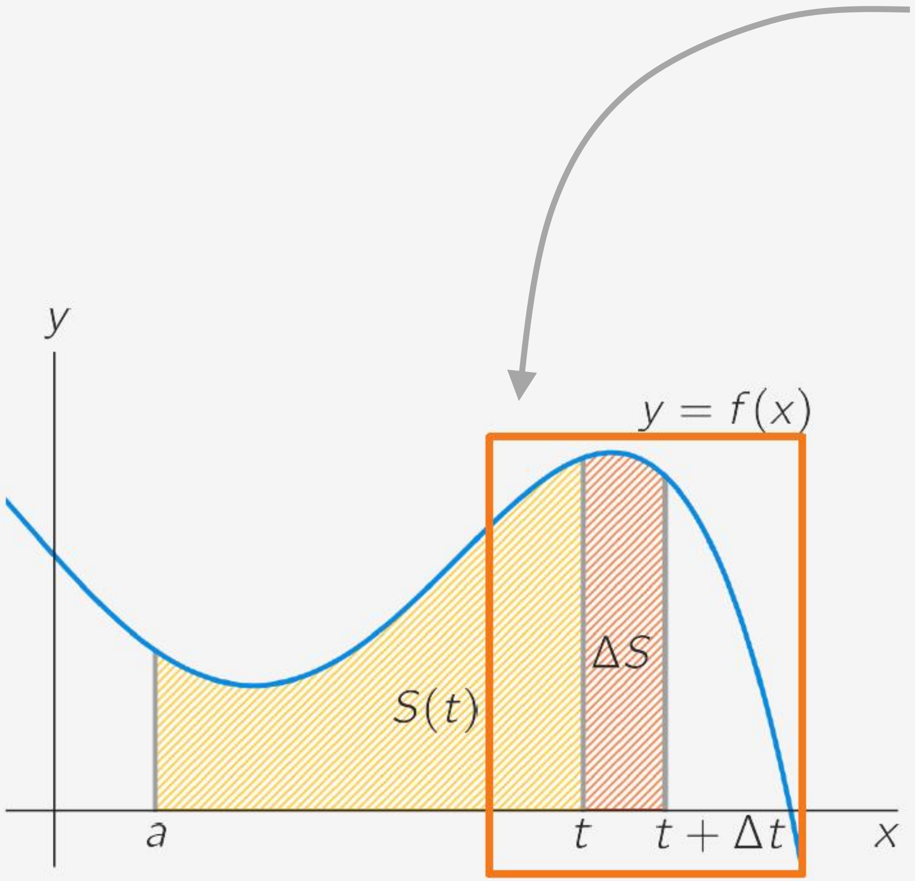

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$$\left.\begin{aligned} S(a) &= \int_a^a f(x)dx = 0 \\ S(b) &= \int_a^b f(x)dx \end{aligned}\right\} \Rightarrow \boxed{\int_a^b f(x)dx = S(b) - S(a)}$$

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Justification for the FTC: general case

★ $S(t) = \int_a^t f(x)dx$ is a primitive of f

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F **any** other primitive of f

$\Longrightarrow S = F + C \qquad (C = \text{constant})$

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$$\int_a^b f(x)dx = S(b) - S(a)$$

$$= F(b) + C - (F(a) + C)$$

$$= F(b) - F(a)$$

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FUNDAMENTAL THEOREM OF CALCULUS