Numerical Differentiation

Ciprian D Coman

Outline

- ❑ Motivation
- ❑ Finite-difference formulae
- ❑ Introduction to error analysis

MOTIVATION

Motivation

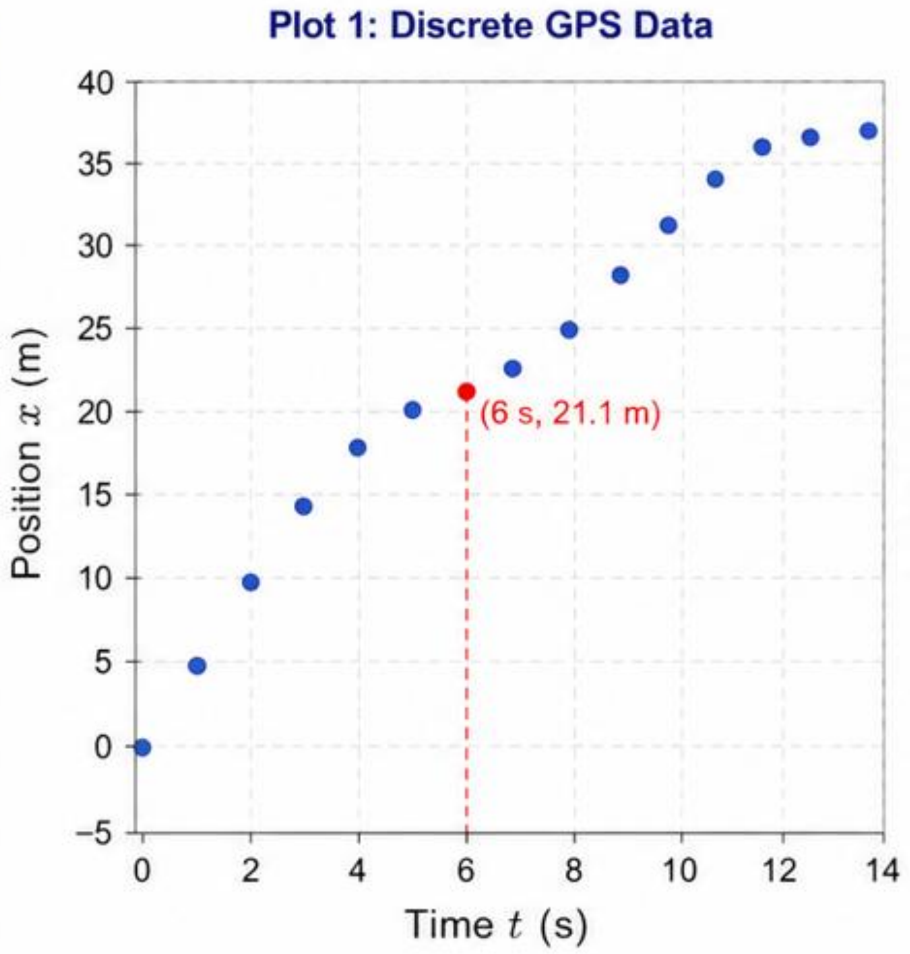
A GPS device records the position of a car **every second** along a road.

How fast was the car travelling at $t = 6\ s$?

GPS Position Data

Time t (s)	Position x (m)
0	0.0
1	4.8
2	9.7
3	14.1
4	17.8
5	20.3
6	21.1
7	22.4
8	24.8
9	27.9
10	31.4
11	34.2
12	36.0
13	36.6
14	36.9

Position is measured along a straight road (meters). Positive is east.



Motivation

A GPS device records the position of a car **every second** along a road.

How fast was the car travelling at $t = 6\ s$?

Speed is the **rate of change** of position:

$$v(t) = \frac{dx}{dt}$$

But we *only* have position measurements at discrete times.

How can we **estimate** $v(6)$ **from** this **data**?

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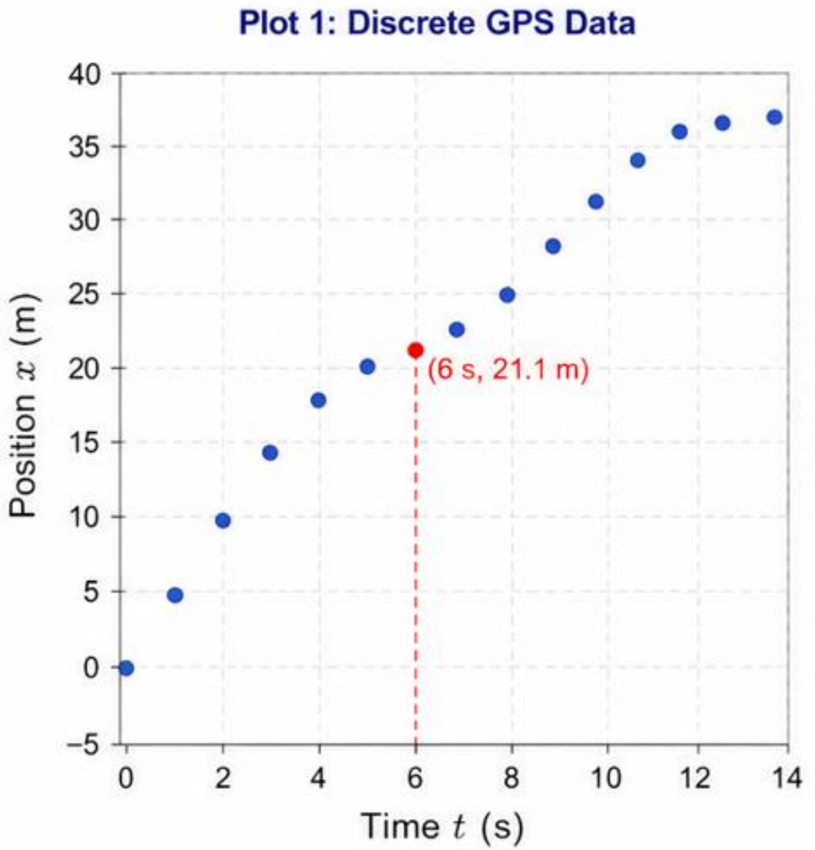

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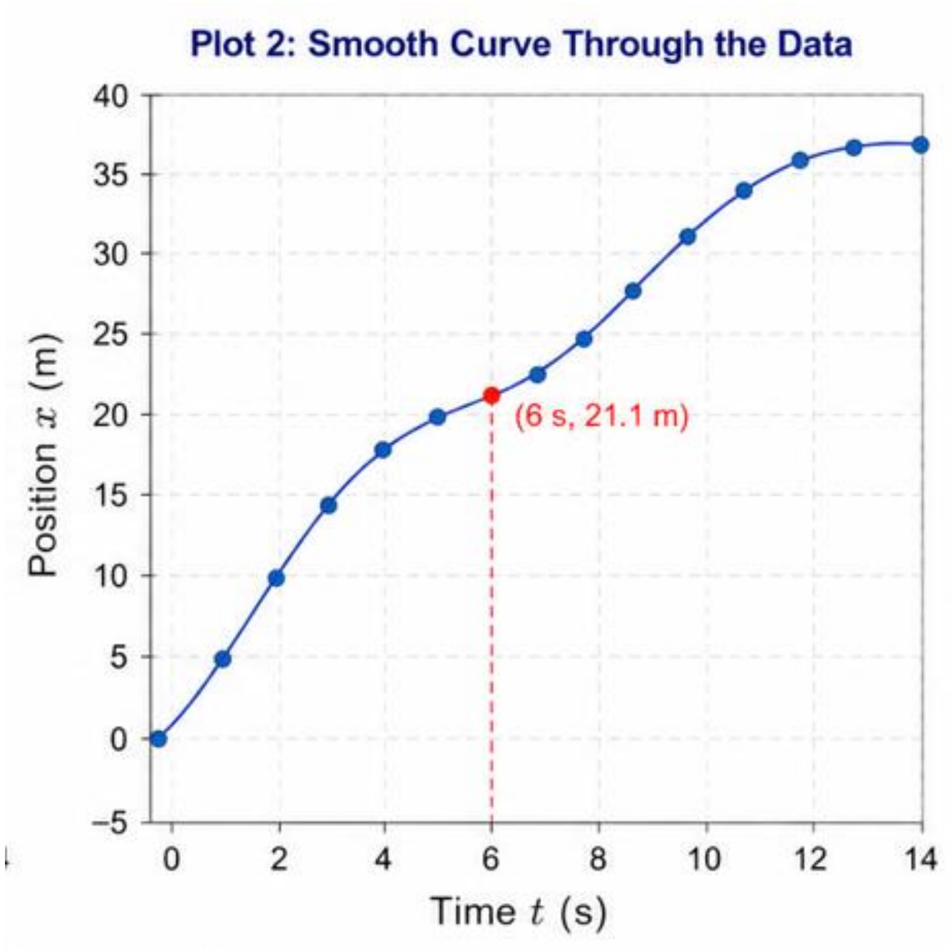
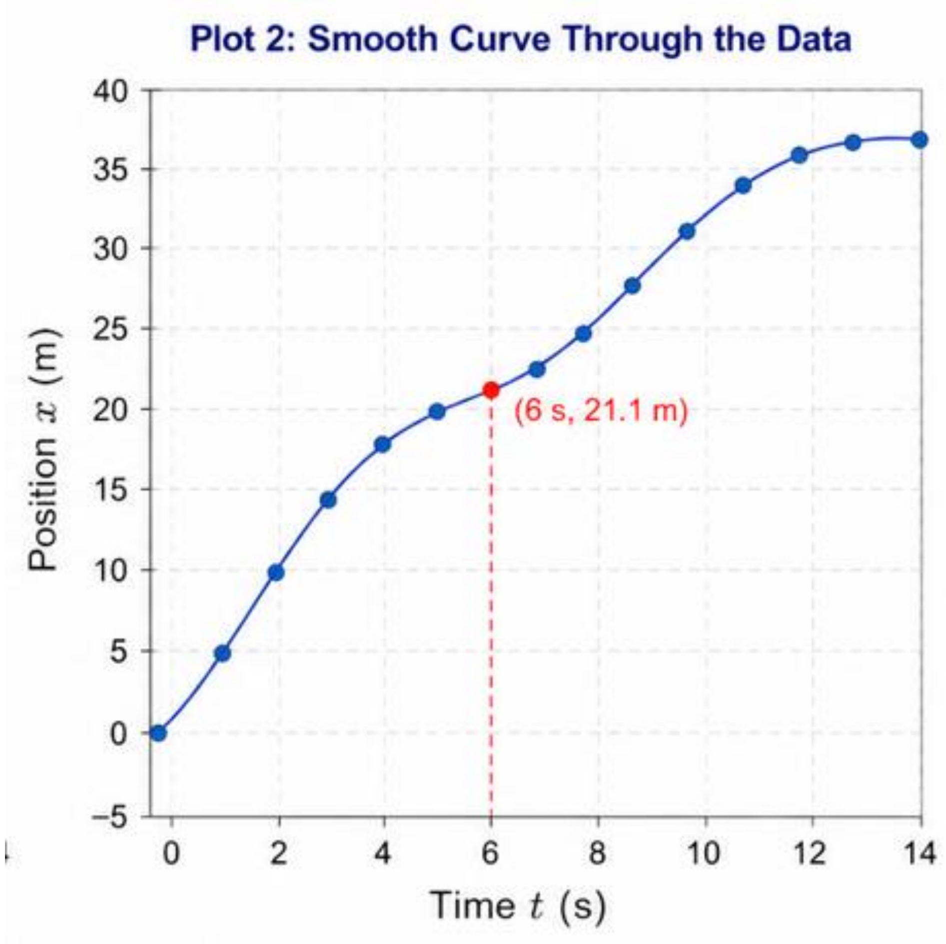


Plot 2: Smooth Curve Through the Data
(6 s, 21.1 m)
Position x (m)
Time t (s)
40
35
30
25
20
15
10
5
0
−5
0
2
4
6
8
10
12
14

Motivation



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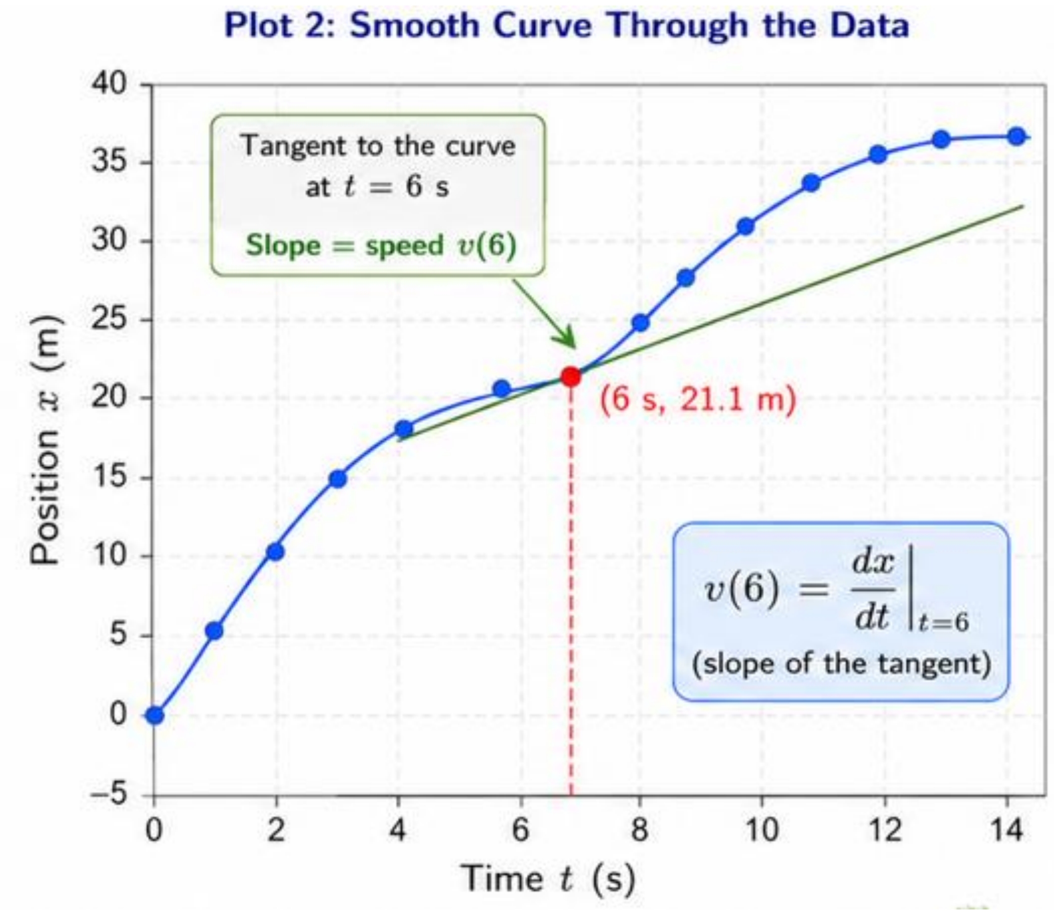
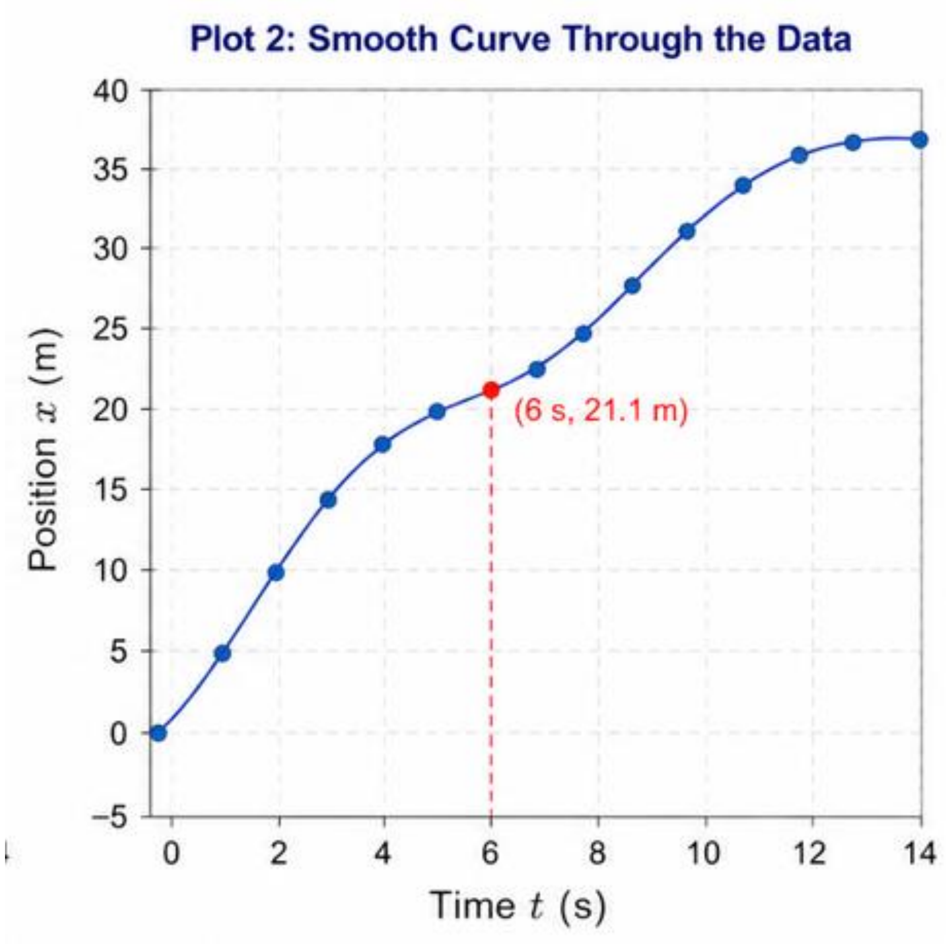


Plot 2: Smooth Curve Through the Data
Tangent to the curve
at $t = 6$ s
Slope = speed $v(6)$
(6 s, 21.1 m)
$v(6) = \left.\frac{dx}{dt}\right|_{t=6}$
(slope of the tangent)
Position x (m)
Time t (s)

Motivation



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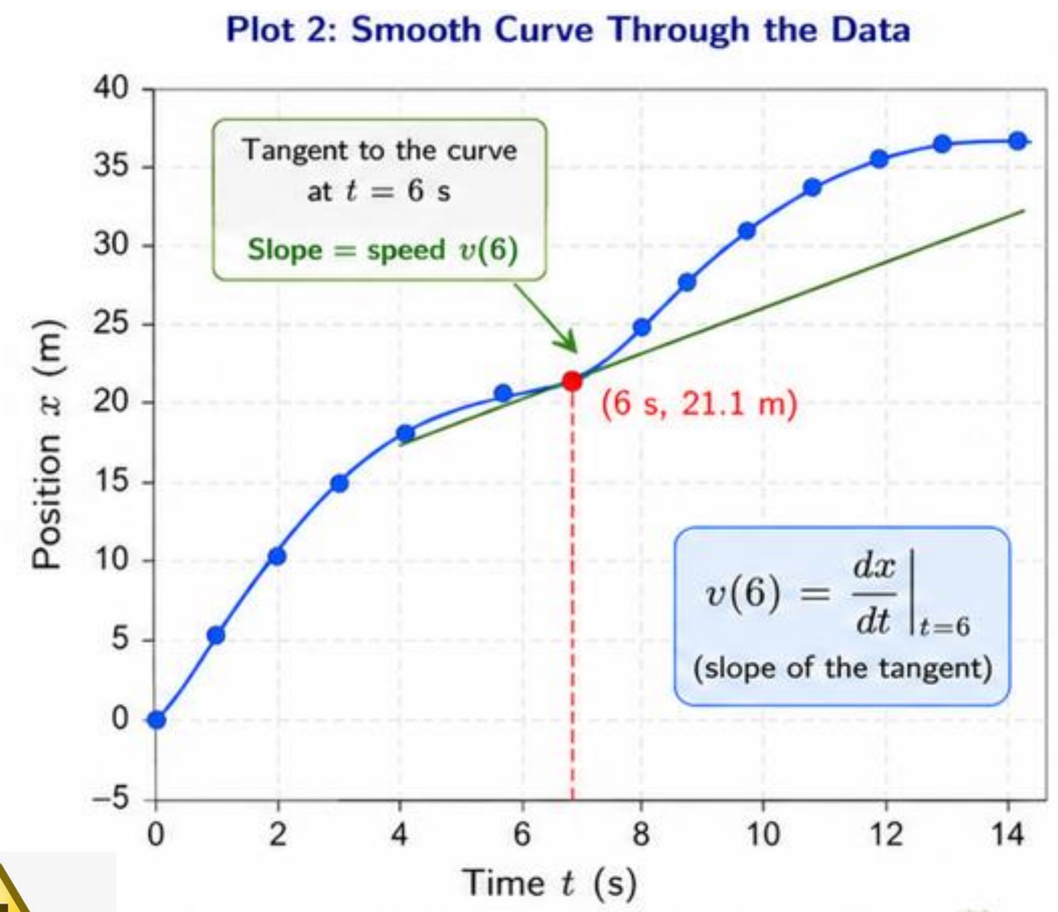

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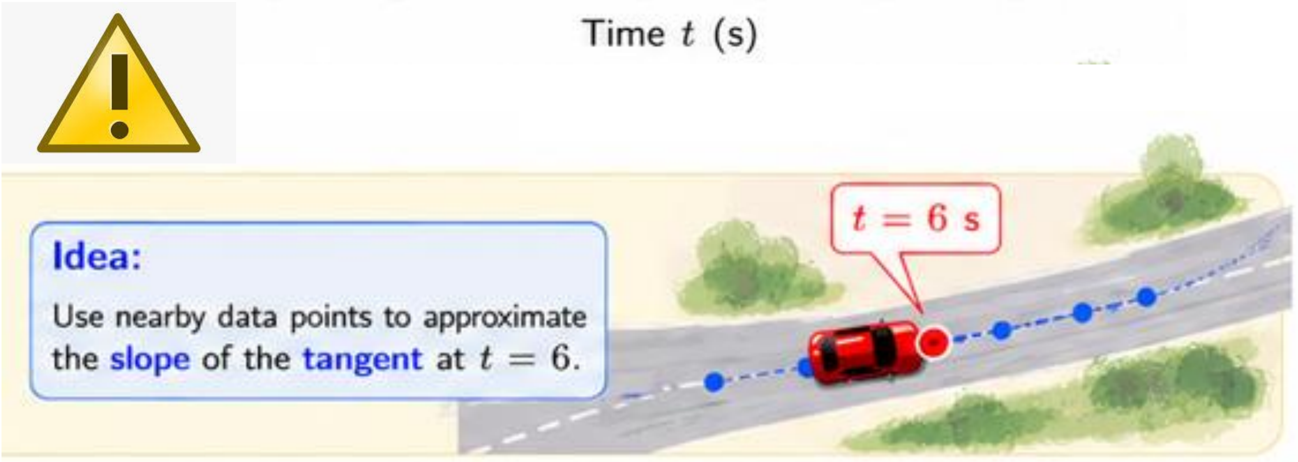
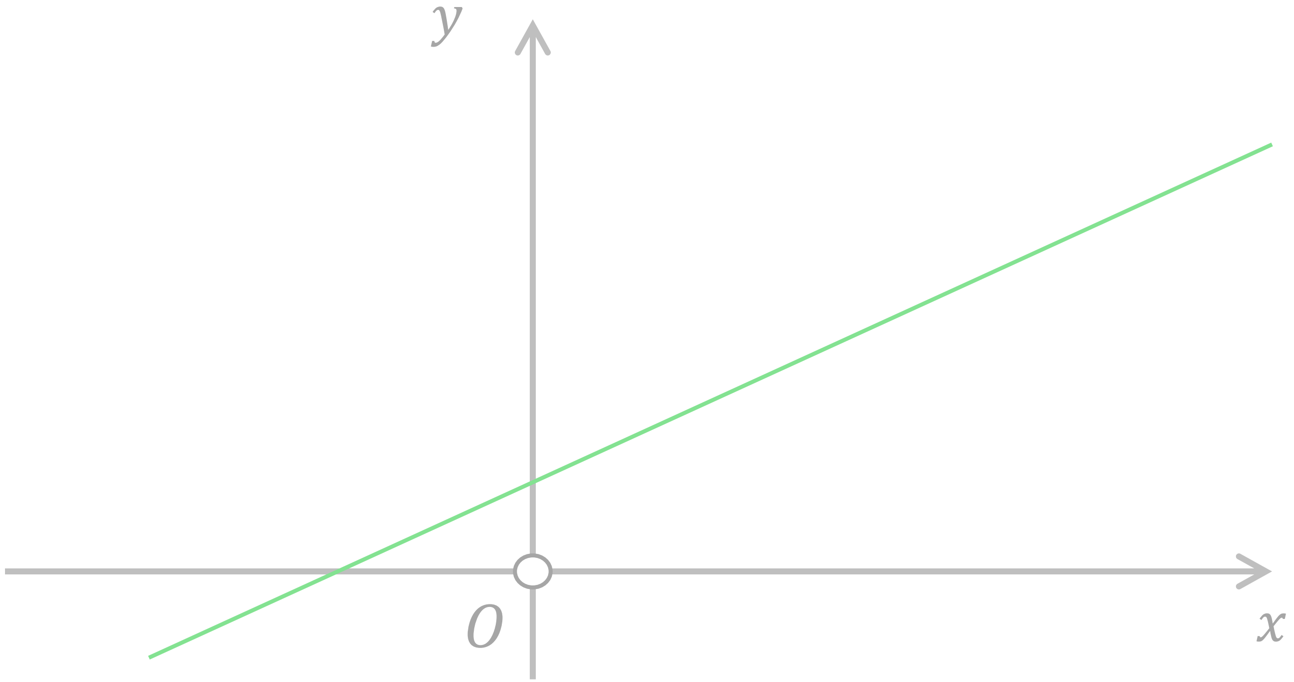


Idea:
Use nearby data points to approximate the slope of the tangent at $t = 6$.
$t = 6$ s

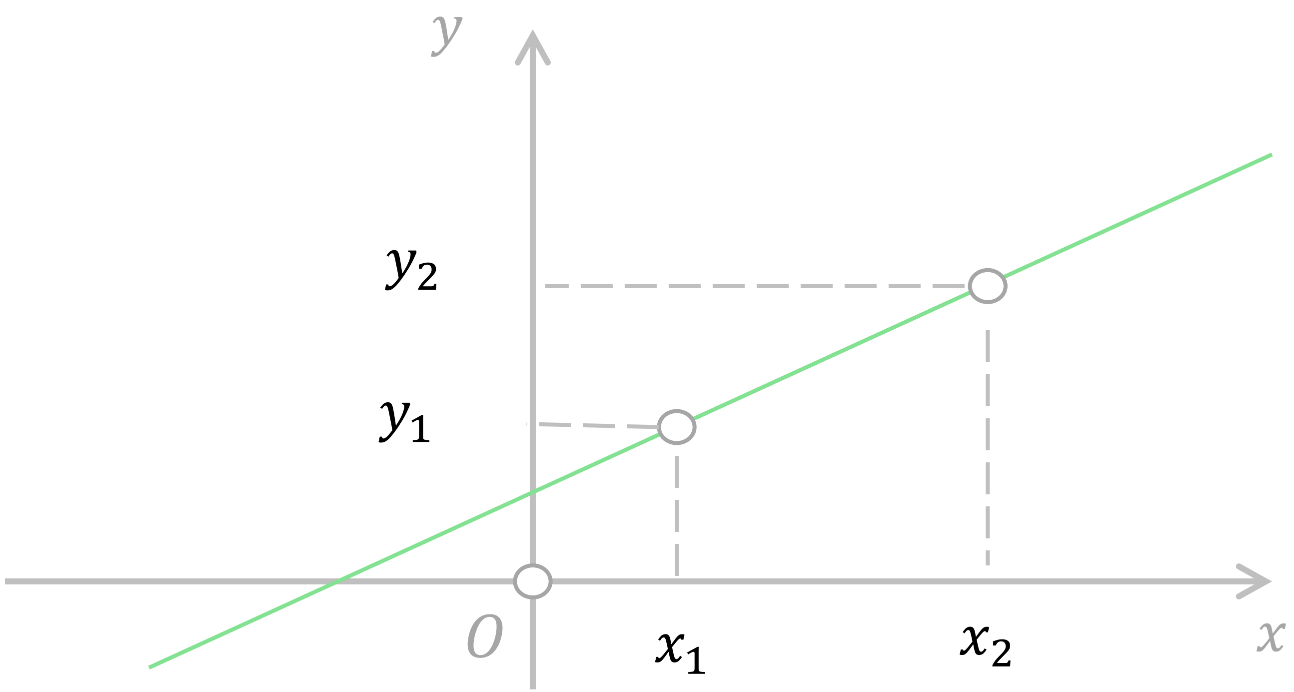
ASIDE: slope of a line



ASIDE: slope of a line

Any two points on the line:

(x_1, y_1) & (x_2, y_2)



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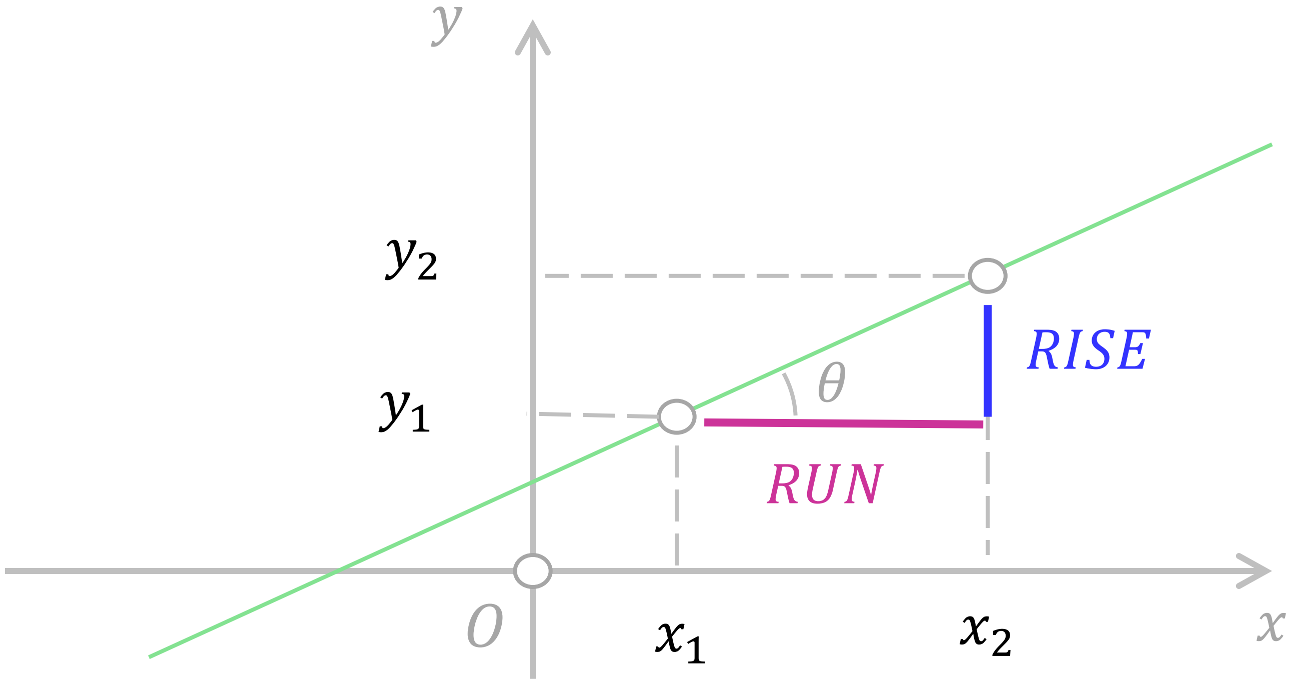
$$SLOPE = \frac{RISE}{RUN}$$



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Any two points on the line:

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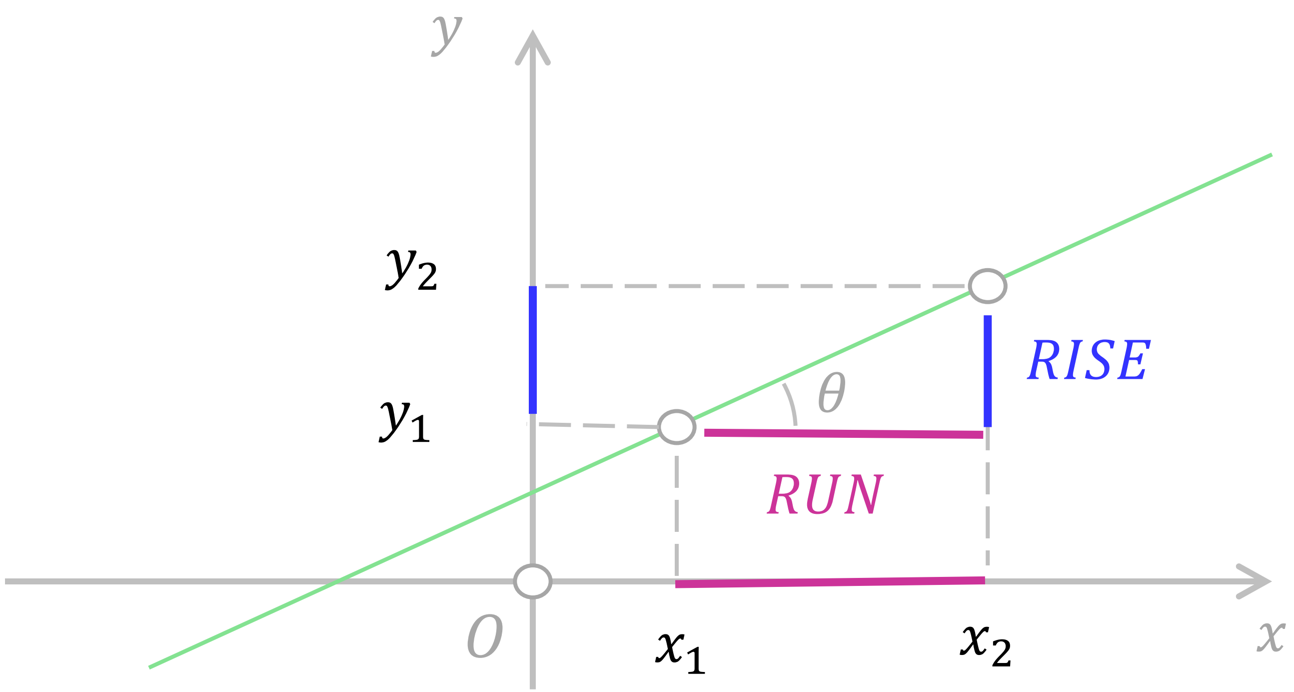
$$SLOPE = \frac{RISE}{RUN} = \frac{y_2 - y_1}{x_2 - x_1}$$



FINITE-DIFFERENCE

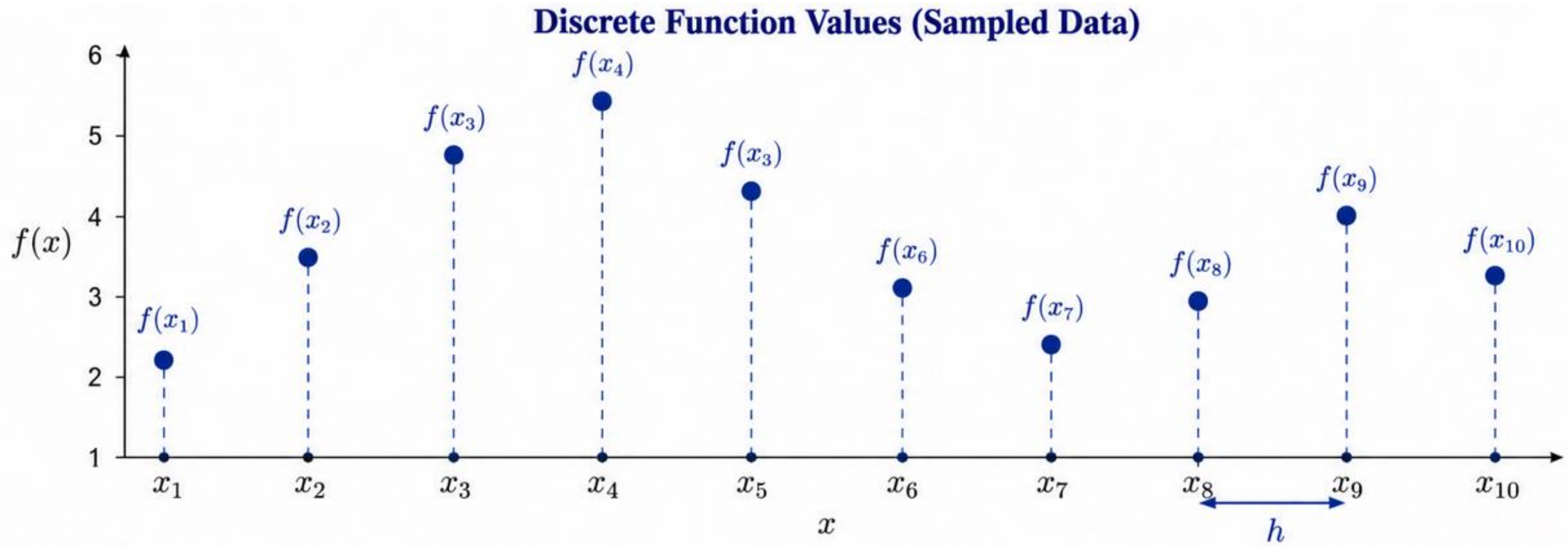
FORMULAE

Setting the stage

x_j = equally spaced points

$f_j \equiv f(x_j)$ = sampled data

$j = 1, 2, 3, \ldots$



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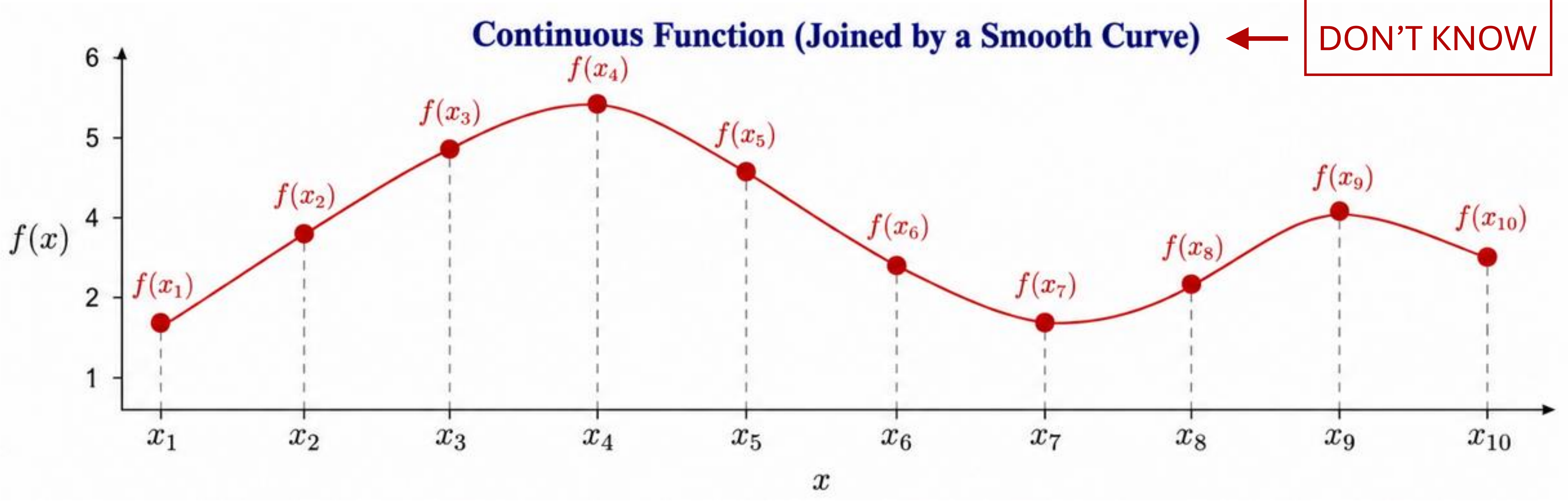
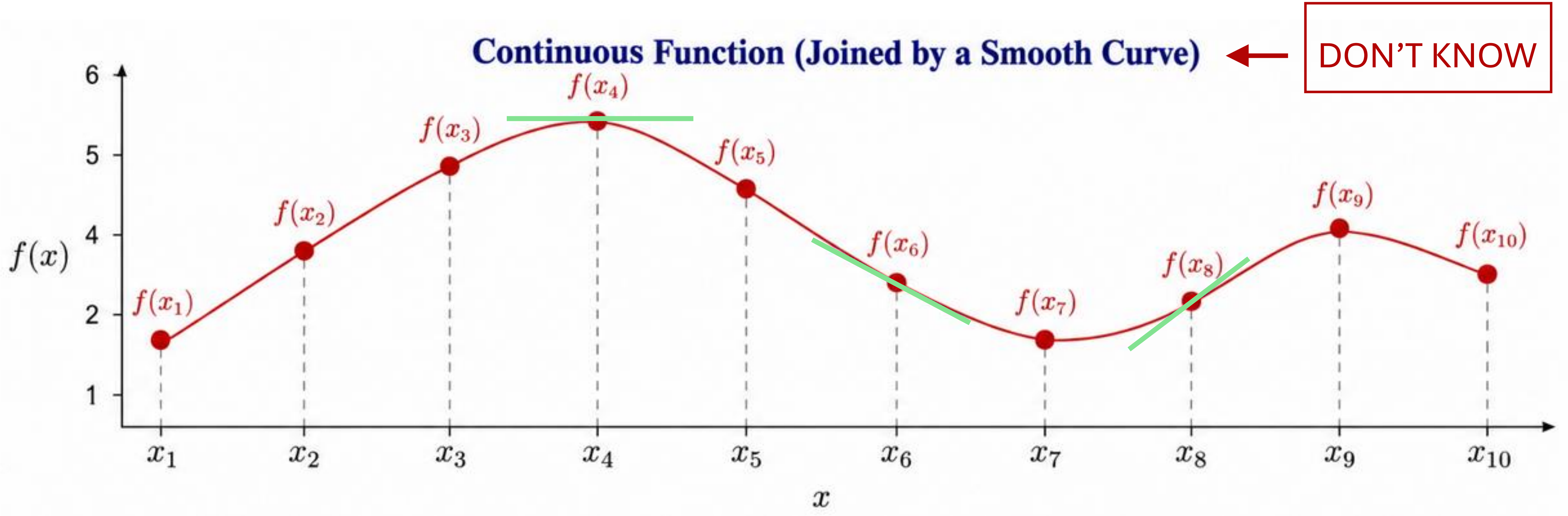


DON'T KNOW

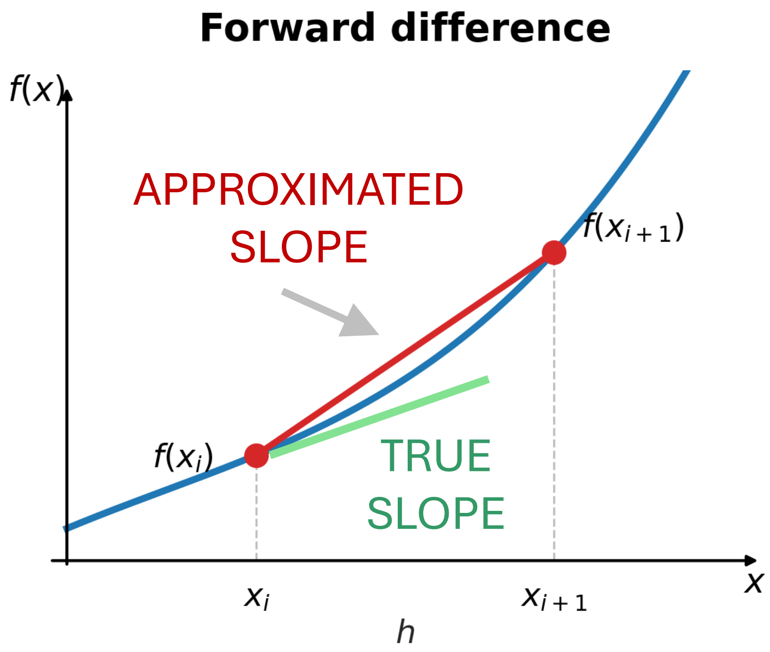
Setting the stage

OUR GOAL:

Want to **approximate** the slopes of the green line segments



Ways to approximate slopes
(finite-difference formulae)



$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{h}$$

"approximately equal to"

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this means that:

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + \underbrace{ERROR\ TERM}_{E_h}$$

ASIDE: "approximately equal to"

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{h}$$

this means that:

$$\underbrace{f'(x_i)}_{\text{TRUE VALUE}} = \underbrace{\frac{f(x_{i+1}) - f(x_i)}{h}}_{\text{PROPOSED APPROXIMATION}} + \underbrace{ERROR\ TERM}_{E_h}$$

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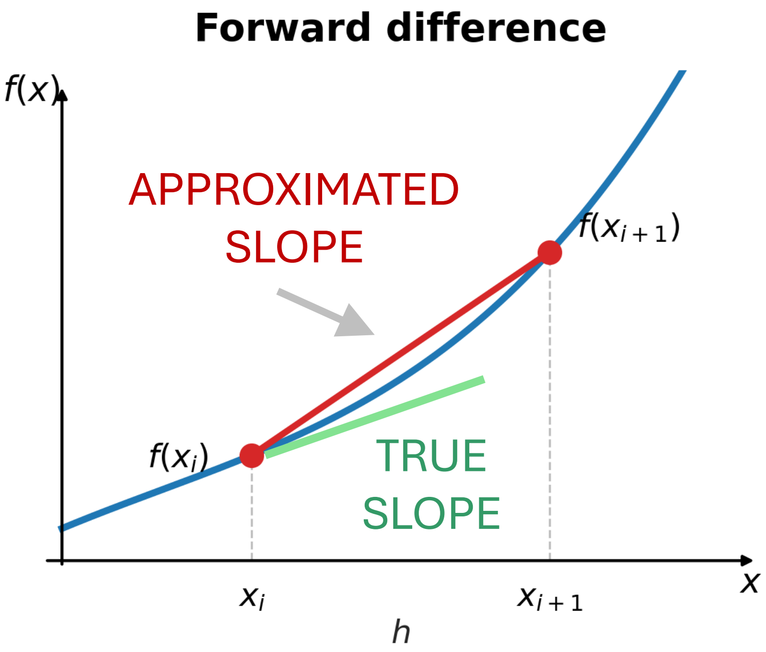

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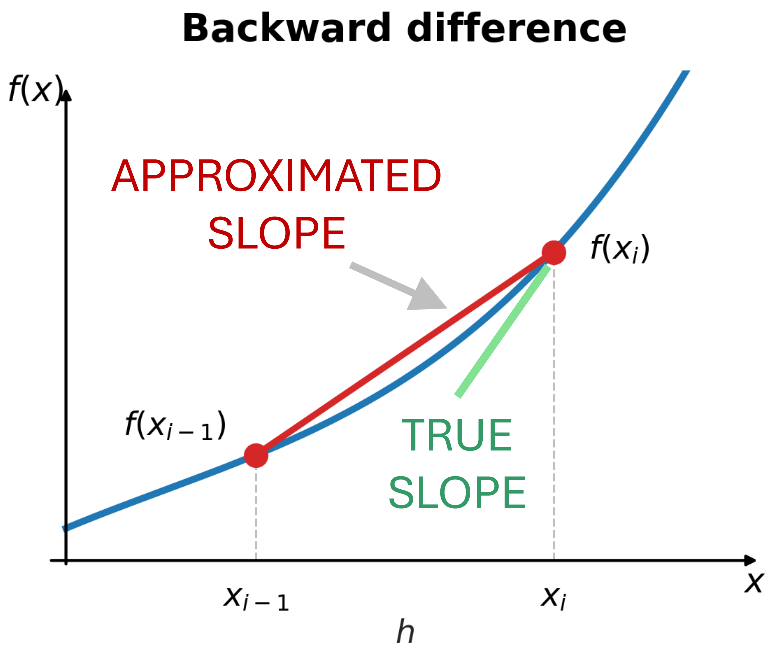

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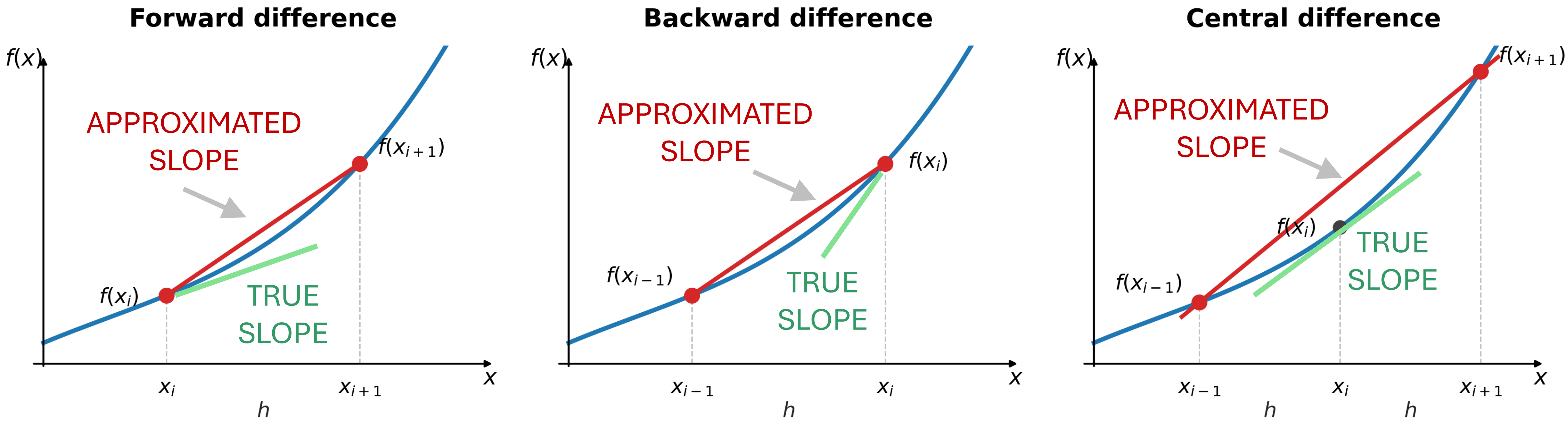

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Example

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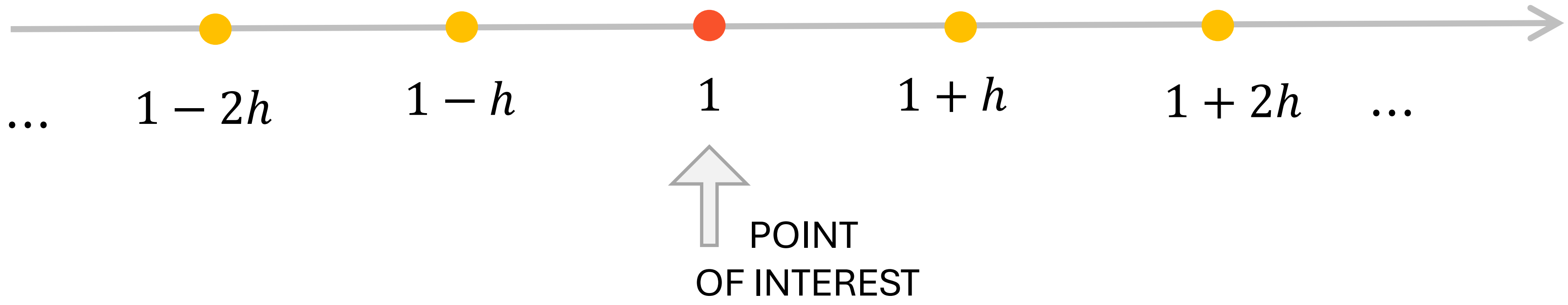

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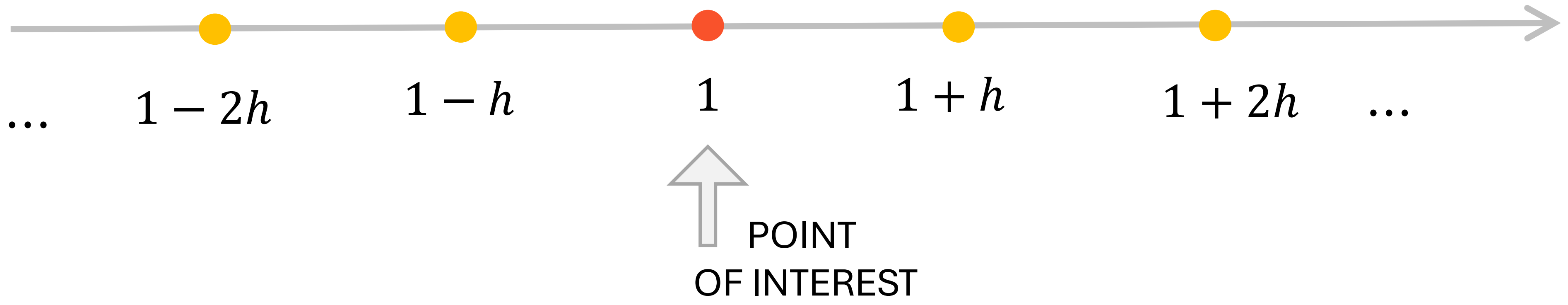

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Step size h	Forward Difference	Backward Difference	Central Difference
0.10	2.85884	2.58679	2.72281
0.05	2.78739	2.65167	2.71941
0.01	2.73192	2.70474	2.71833

Try it yourself
(3 minute activity)

The values of a function are given below.

x	1.9	2.0	2.1
$f(x)$	3.61	4.00	4.41

Estimate $f'(2)$ using:

1. the forward difference;
2. the backward difference;
3. the central difference.

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Estimate $f'(2)$ using:

1. the forward difference;
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Answers:

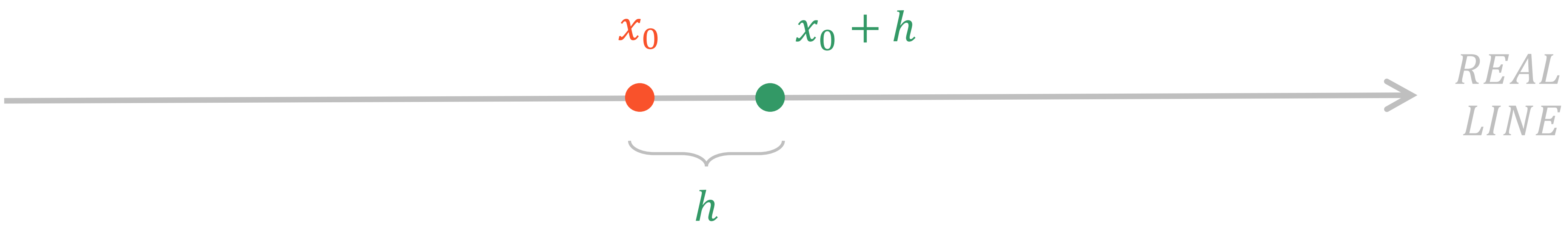
$$f'(2) \approx \frac{4.41 - 4.00}{0.1} = 4.1$$

$$f'(2) \approx \frac{4.00 - 3.61}{0.1} = 3.9$$

$$f'(2) \approx \frac{4.41 - 3.61}{0.2} = 4.0$$

ERROR ANALYSIS

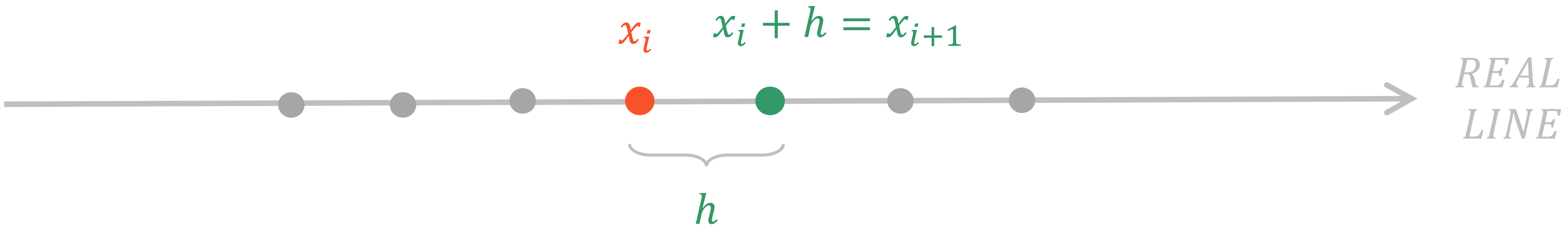
Taylor's formula



$$f(x_0 + h) = f(x_0) + \frac{h}{1!} f'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \cdots$$

for any h small

Estimating the ERROR
(Forward Difference)



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Set $x_0 = x_i$ and note that $x_0 + h = x_i + h = x_{i+1}$

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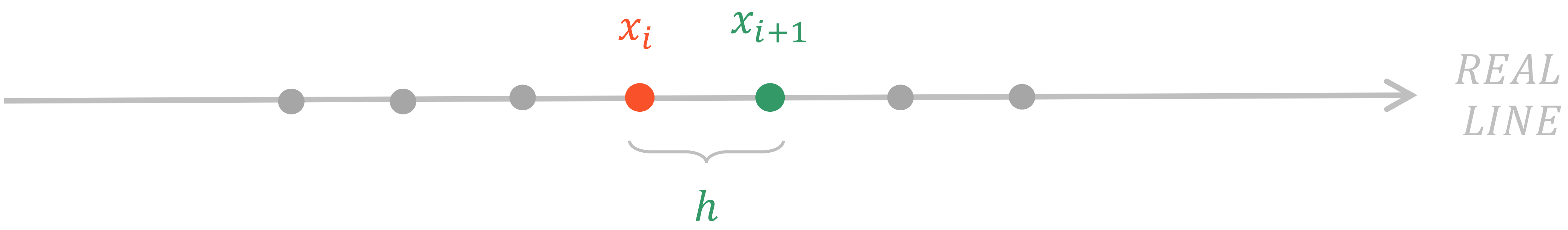

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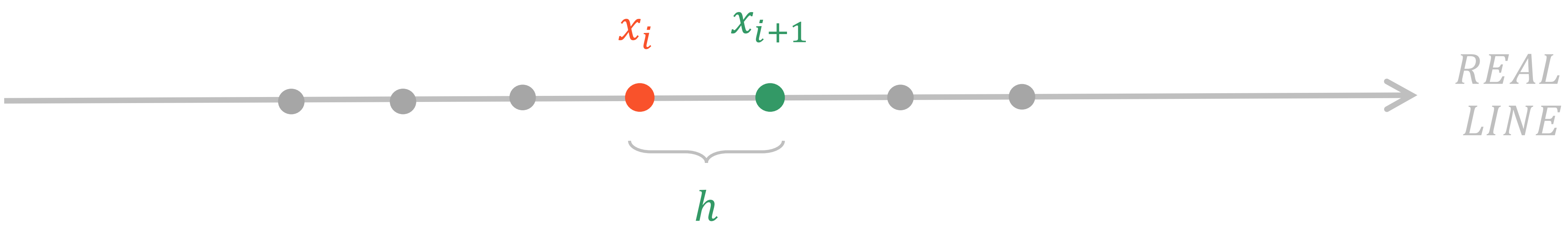

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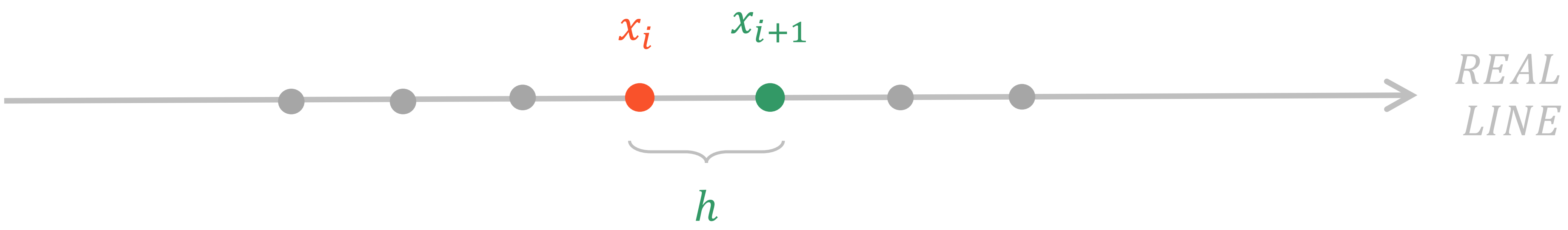
SOLVE FOR THIS

Estimating the ERROR
(Forward Difference)



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$$\Rightarrow \quad f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{h}{2!} f''(x_i) - \frac{h^2}{3!} f'''(x_i) - \cdots$$

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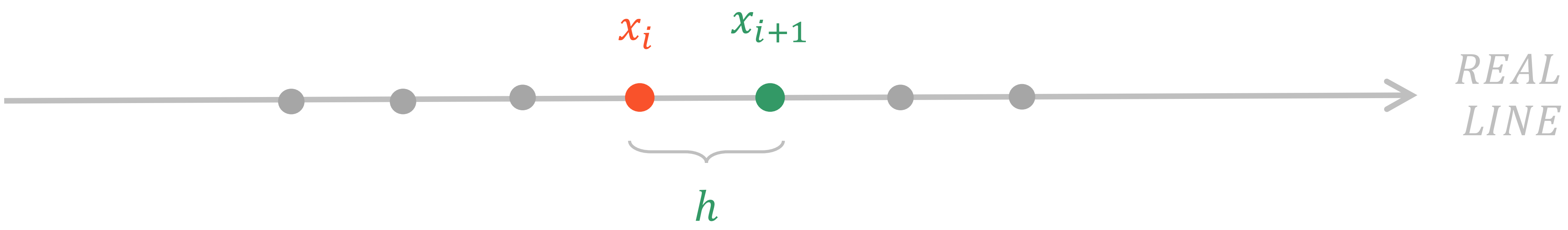

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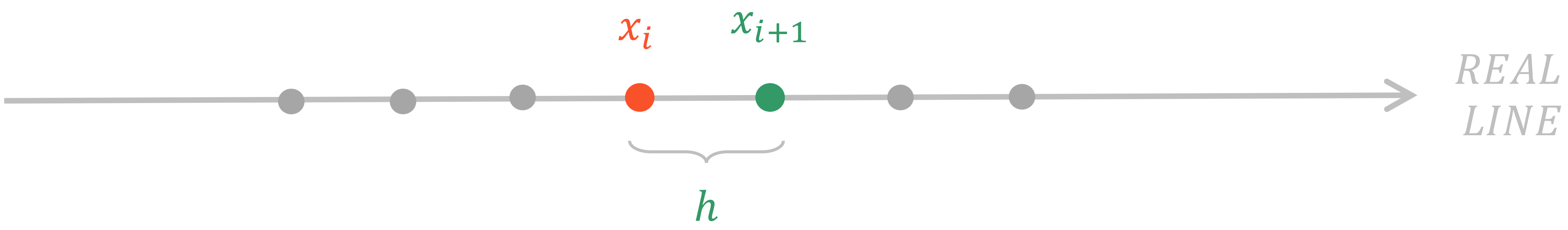
ERROR TERM (E_h) → also known as **truncation error**

Estimating the ERROR
(Forward Difference)



$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + E_h$$

$$E_h = -\frac{h}{2!}f''(x_i) - \frac{h^2}{3!}f'''(x_i) - \cdots$$

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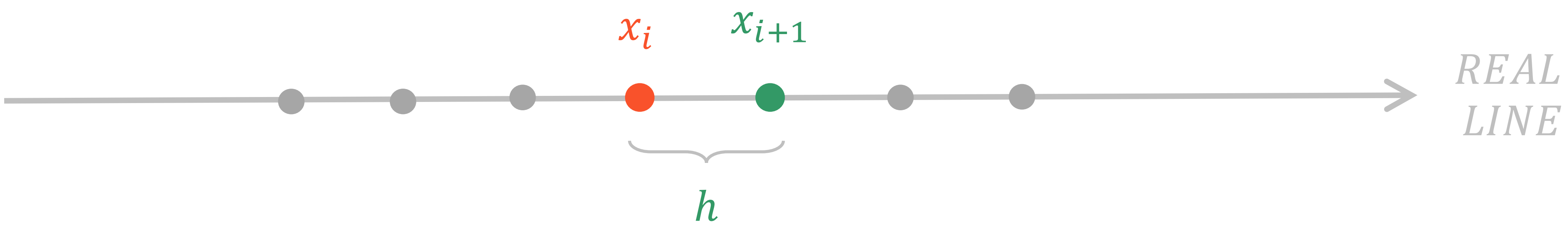
(Forward Difference)



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$$= h\left[-\frac{1}{2!}f''(x_i) - \frac{h}{3!}f'''(x_i) - \cdots\right]$$

$$= h \times constant = Ch$$

Estimating the ERROR
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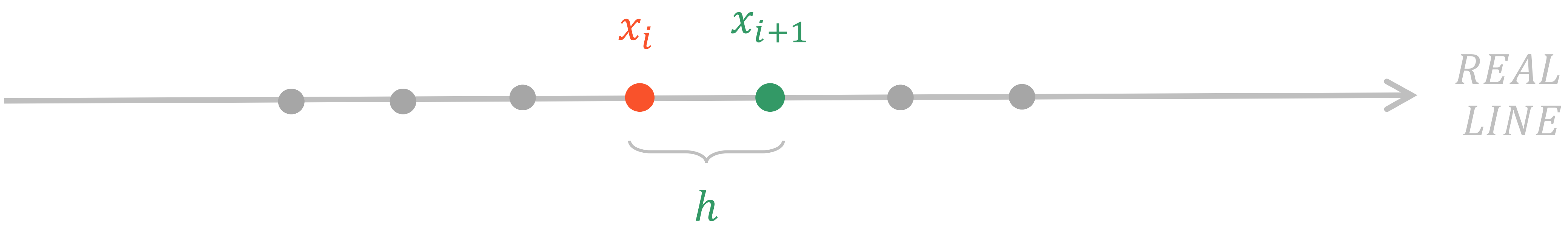

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We say that the **truncation error** in the Forward-Difference formula is of order h.

This is abbreviated as $E_h = O(h)$

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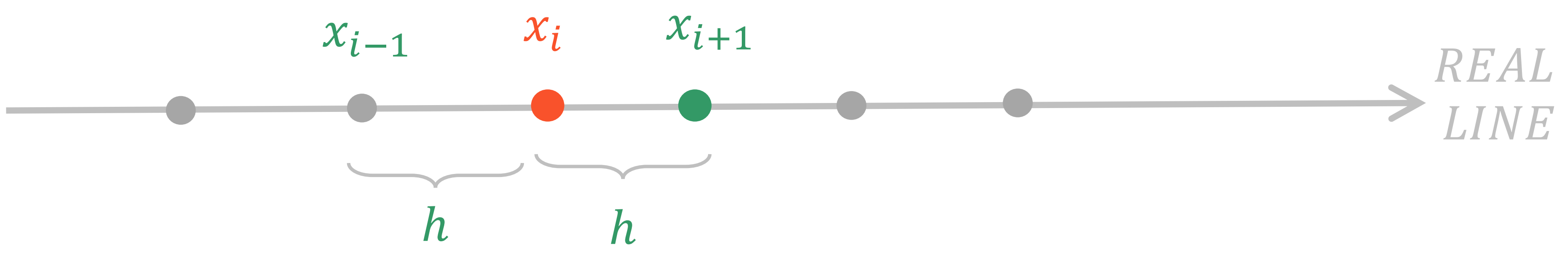
Other error estimates

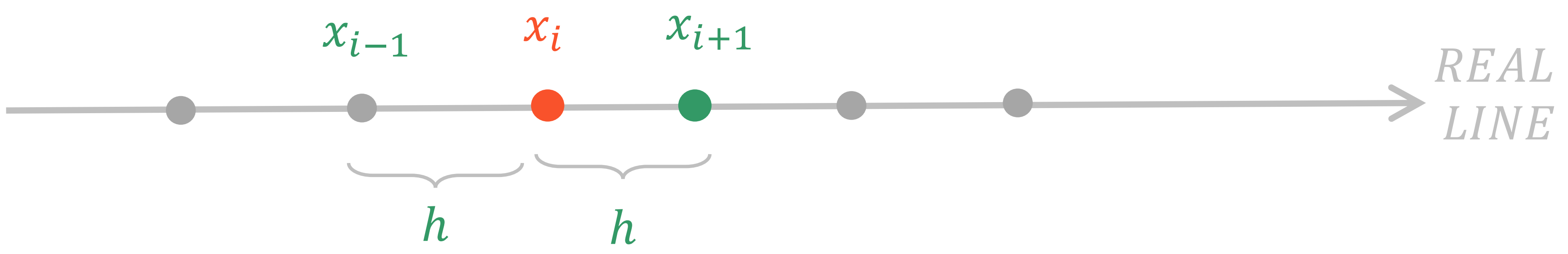


$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h} + O(h)$$

ERROR of order h

Backward Difference

Other error estimates



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ERROR of order h

Backward Difference

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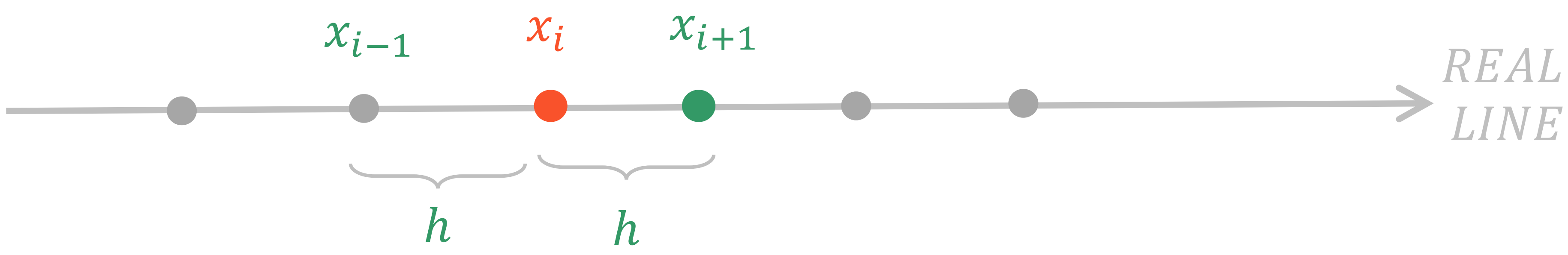
ERROR of order h^2

Central Difference

Other error estimates



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ERROR
of order h

Backward Difference

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ERROR
of order h^2

Central Difference

this means the error term is **proportional to** h^2

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Accuracy: $O(h)$

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