Algebraic & **Geometric** Multiplicities

The main definition:

Let $\boldsymbol{A} \in \mathcal{M}_{n\times n}(\mathbb{R})$

The number λ (real or complex) is called an **eigenvalue** of $\boldsymbol{A}$ if there is a non-zero vector $\boldsymbol{u}$ such that

$$\boldsymbol{Au} = \lambda \boldsymbol{u}\,, \qquad (\boldsymbol{u} \neq 0)$$

The vector $\boldsymbol{u}$ is called an **eigenvector** of $\boldsymbol{A}$ corresponding to the eigenvalue λ

$$\boldsymbol{Au} = \lambda \boldsymbol{u} \Rightarrow \boldsymbol{Au} - \lambda(\boldsymbol{I}_n \boldsymbol{u}) = 0 \Rightarrow \boldsymbol{Au} - (\lambda \boldsymbol{I}_n)\boldsymbol{u} = 0$$

$$\Rightarrow (\boldsymbol{A} - \lambda \boldsymbol{I}_n)\boldsymbol{u} = 0$$

$$\Rightarrow \boldsymbol{A} - \lambda \boldsymbol{I}_n \in \mathcal{M}_{n\times n}(\mathbb{R}) \text{ is singular} \qquad (1)$$

Remember: $\boldsymbol{B} \in \mathcal{M}_{n\times n}(\mathbb{R})$ is non-singular $\Leftrightarrow$ $\det(\boldsymbol{B}) \neq 0$

or, equivalently, $\boldsymbol{B}$ is singular $\Leftrightarrow$ $\det(\boldsymbol{B}) = 0$

Use this result in (1), with $\boldsymbol{B} \rightarrow \boldsymbol{A} - \lambda \boldsymbol{I}_n$:

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}_n) = 0$$

the *eigenvalues* are found by solving this equation characteristic polynomial → $p_A(\lambda) := \det(\boldsymbol{A} - \lambda \boldsymbol{I}_n)$

characteristic equation → $p_A(\lambda) = 0$

Particular case: $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

2×2 matrix

$$\boldsymbol{A} - \lambda \boldsymbol{I}_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix}$$

$$p_A(\lambda) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21}$$

$$\Rightarrow p_A(\lambda) = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})$$

polynomial of degree 2

In general: if $\boldsymbol{A} \in \mathcal{M}_{n\times n}(\mathbb{R})$ then $p_A(\lambda)$ is a polynomial of degree n:

$$p_A(\lambda) = p_n\lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_1\lambda + p_0\,, \qquad (p_j \in \mathbb{R})$$

$$\Rightarrow p_A(\lambda) = p_n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_{n-1})(\lambda - \lambda_n)$$

Aside (Fundamental Theorem of Algebra)**:** Any polynomial of degree n with real (or complex) coefficients has **exactly** *n* roots **(counting repeated roots as well)**

Algebraic multiplicity of the eigenvalue λ_j ⇒ the number of times the factor $(\lambda - \lambda_j)$ appears in $p_A(\lambda)$

Example #1: $\boldsymbol{A} = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$

$$p_A(\lambda) = \det(\boldsymbol{A} - \lambda \boldsymbol{I}_2) = \begin{vmatrix} 4-\lambda & 1 \\ 0 & 4-\lambda \end{vmatrix} = (\lambda - 4)^2$$

$\Rightarrow \quad \lambda = 4$ is an eigenvalue of algebraic multiplicity 2

Example #2:

$$p_A(\lambda) = \det(\boldsymbol{A} - \lambda \boldsymbol{I}_3) = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix}$$

$$\boldsymbol{A} = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

$$= -\lambda^3 + 6\lambda^2 + 15\lambda + 8 = -(\lambda + 1)^2(\lambda - 8)$$

$\Rightarrow \begin{cases} \lambda = 8 & \text{has algebraic multiplicity 1} \\ \lambda = -1 & \text{has algebraic multiplicity 2} \end{cases}$

Geometric multiplicity

$$\boldsymbol{A}\boldsymbol{u} = \lambda \boldsymbol{u}\,, \quad (\boldsymbol{u} \neq 0) \quad \Longleftrightarrow \quad (\boldsymbol{A} - \lambda \boldsymbol{I}_n)\boldsymbol{u} = 0$$

eigenvalue: λ; eigenvector: $\boldsymbol{u}$

Let's consider the set:

$$\mathcal{E}_\lambda := \left\{ \boldsymbol{u} \in \mathbb{C}^n \;\middle|\; (\boldsymbol{A} - \lambda \boldsymbol{I}_n)\boldsymbol{u} = 0 \right\}$$

Observations:

- if $\alpha \in \mathbb{C}$ and $\boldsymbol{u} \in \mathcal{E}_\lambda \Rightarrow \alpha \boldsymbol{u} \in \mathcal{E}_\lambda$
- if $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathcal{E}_\lambda \Rightarrow \boldsymbol{u}_1 + \boldsymbol{u}_2 \in \mathcal{E}_\lambda$

Conclusion: $\mathcal{E}_\lambda$ is a linear subspace of $\mathbb{C}^n$

the eigenspace of A corresponding to λ

$$p_A(\lambda) = p_n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_{n-1})(\lambda - \lambda_n)$$

Geometric multiplicity of the eigenvalue λ_j ⇒ dimension of the eigenspace $\mathcal{E}_{\lambda_j}$

Example #1: $\boldsymbol{A} = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$

$\lambda = 4$, algebraic multiplicity = 2

$$(\boldsymbol{A} - 4\boldsymbol{I}_2)\boldsymbol{u} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_2 \\ 0 \end{bmatrix}$$

$$(\boldsymbol{A} - 4\boldsymbol{I}_2)\boldsymbol{u} = \boldsymbol{0} \Leftrightarrow \begin{bmatrix} u_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \boldsymbol{u} = \begin{bmatrix} u_1 \\ 0 \end{bmatrix} = u_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

so, if $\boldsymbol{u} \in \mathcal{E}_4 \Rightarrow \boldsymbol{u} = \text{const.} \times \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\Rightarrow \mathcal{E}_4 = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} \Rightarrow \dim\left(\mathcal{E}_4\right) = 1$$

⇒ geometric multiplicity = 1

Example #2: $\boldsymbol{A} = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

$$\begin{cases} \lambda_1 = 8 & \text{has algebraic multiplicity 1} \\ \lambda_2 = -1 & \text{has algebraic multiplicity 2} \end{cases}$$

$\lambda_2 = -1$

$$(\boldsymbol{A}+\boldsymbol{I}_3)\boldsymbol{u} = \boldsymbol{0} \Leftrightarrow \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow 2u_1 + u_2 + 2u_3 = 0$$

$$\Rightarrow u_2 = -2u_1 - 2u_3$$

$u_1 \in \mathbb{R}$
$u_3 \in \mathbb{R}$
arbitrary

$$\text{so, if } \boldsymbol{u} \in \mathcal{E}_{-1} \Rightarrow \boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ -2u_1 - 2u_3 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ -2u_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2u_3 \\ u_3 \end{bmatrix}$$

$$\Rightarrow \boldsymbol{u} = u_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + u_3 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \Rightarrow \mathcal{E}_{-1} = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\} \Rightarrow \dim\left(\mathcal{E}_{-1}\right) = 2$$

$\Rightarrow$ geometric multiplicity of λ_2 is 2

$\lambda_1 = 8$

$$(\boldsymbol{A} - 8\boldsymbol{I}_3)\boldsymbol{u} = \boldsymbol{0} \Leftrightarrow \begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (*)$$

$$\left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 2 & -8 & 2 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right] \underset{\substack{R_2 \to R_2 + 4R_1 \\ R_3 \to R_3 - R_1}}{\sim} \left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ -18 & 0 & 18 & 0 \\ 9 & 0 & -9 & 0 \end{array}\right] \underset{R_2 \to R_2/18}{\sim} \left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ -1 & 0 & 1 & 0 \\ 9 & 0 & -9 & 0 \end{array}\right] \underset{\substack{R_3 \to R_3 + 9R_2 \\ R_1 \to R_1 - 5R_2}}{\sim} \left[\begin{array}{ccc|c} 0 & 2 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$(*) \Leftrightarrow \begin{bmatrix} 0 & 2 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} u_3 = 2u_2 \\ u_1 = u_3 \end{cases}$$

$$\text{so, if } \boldsymbol{u} \in \mathcal{E}_8 \Rightarrow \boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 2u_2 \\ u_2 \\ 2u_2 \end{bmatrix} = u_2 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \mathcal{E}_8 = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\} \Rightarrow \dim(\mathcal{E}_8) = 1$$

$u_2 \in \mathbb{R}$ arbitrary

$\Rightarrow$ geometric multiplicity of λ_1 is 1

	Algebraic Multiplicity	Geometric Multiplicity
Example #1 $\lambda_1 = 4$	**2**	**1**
Example #2 $\lambda_1 = 8$ $\lambda_2 = -1$	**1** **2**	**1** **2**

Theorem:

Let λ be an eigenvalue of $\boldsymbol{A} \in \mathcal{M}_{n\times n}(\mathbb{R})$. Then

$1 \leq$ geometric multiplicity of $\lambda \leq$ algebraic multiplicity of λ