

Algebraic & Geometric Multiplicities

The main definition:

Let $A \in \mathcal{M}_{n \times n}(\mathbb{R})$

The number λ (real or complex) is called an **eigenvalue** of A if there is a non-zero vector u such that

$$Au = \lambda u, \quad (u \neq 0)$$

The vector u is called an **eigenvector** of A corresponding to the eigenvalue λ

$$Au = \lambda u \Rightarrow Au - \lambda(I_n u) = 0 \Rightarrow Au - (\lambda I_n)u = 0$$

$$\Rightarrow (A - \lambda I_n)u = 0$$

$$\Rightarrow A - \lambda I_n \in \mathcal{M}_{n \times n}(\mathbb{R}) \text{ is singular} \tag{1}$$

Remember: $B \in \mathcal{M}_{n \times n}(\mathbb{R})$ is non-singular $\Leftrightarrow \det(B) \neq 0$

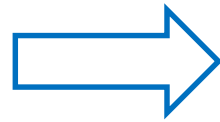
or, equivalently, B is singular $\Leftrightarrow \det(B) = 0$

Use this result in (1), with $B \rightarrow A - \lambda I_n$:

$$\det(A - \lambda I_n) = 0$$

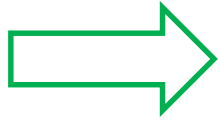
the *eigenvalues*
are found by
solving this
equation

characteristic
polynomial



$$p_A(\lambda) := \det(\mathbf{A} - \lambda \mathbf{I}_n)$$

characteristic
equation



$$p_A(\lambda) = 0$$

Particular case: $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ 2×2 matrix

$$\mathbf{A} - \lambda \mathbf{I}_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix}$$

$$p_A(\lambda) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21}$$

$$\Rightarrow p_A(\lambda) = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})$$



polynomial of degree 2

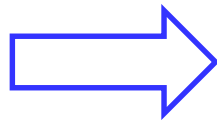
In general: if $\mathbf{A} \in \mathcal{M}_{n \times n}(\mathbb{R})$ then $p_{\mathbf{A}}(\lambda)$ is a polynomial of degree n :

$$p_{\mathbf{A}}(\lambda) = p_n \lambda^n + p_{n-1} \lambda^{n-1} + \cdots + p_1 \lambda + p_0, \quad (p_j \in \mathbb{R})$$

$$\Rightarrow p_{\mathbf{A}}(\lambda) = p_n (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_{n-1})(\lambda - \lambda_n)$$

Aside (Fundamental Theorem of Algebra): Any polynomial of degree n with real (or complex) coefficients has **exactly** n roots (**counting repeated roots as well**)

Algebraic multiplicity
of the eigenvalue λ_j



the number of times the factor
 $(\lambda - \lambda_j)$ appears in $p_{\mathbf{A}}(\lambda)$

Example #1: $\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$

$$p_A(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}_2) = \begin{vmatrix} 4 - \lambda & 1 \\ 0 & 4 - \lambda \end{vmatrix} = (\lambda - 4)^2$$

$\Rightarrow \lambda = 4$ is an eigenvalue of algebraic multiplicity 2

Example #2:

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

$$\begin{aligned} p_A(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}_3) &= \begin{vmatrix} 3 - \lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3 - \lambda \end{vmatrix} \\ &= -\lambda^3 + 6\lambda^2 + 15\lambda + 8 = -(\lambda + 1)^2(\lambda - 8) \end{aligned}$$

$\Rightarrow \begin{cases} \lambda = 8 & \text{has algebraic multiplicity 1} \\ \lambda = -1 & \text{has algebraic multiplicity 2} \end{cases}$

Geometric multiplicity

$$Au = \lambda u, \quad (u \neq 0) \quad \longleftrightarrow \quad (A - \lambda I_n)u = 0$$

eigenvalue
↓
↑
eigenvector

Let's consider the set:

$$\mathcal{E}_\lambda := \left\{ u \in \mathbb{C}^n \mid (A - \lambda I_n)u = 0 \right\}$$

Observations:

if $\alpha \in \mathbb{C}$ and $u \in \mathcal{E}_\lambda \Rightarrow \alpha u \in \mathcal{E}_\lambda$

if $u_1, u_2 \in \mathcal{E}_\lambda \Rightarrow u_1 + u_2 \in \mathcal{E}_\lambda$

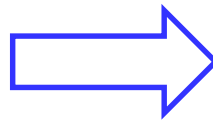
Conclusion:

\mathcal{E}_λ is a linear subspace of \mathbb{C}^n

the eigenspace of A corresponding to λ

$$p_A(\lambda) = p_n(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_{n-1})(\lambda - \lambda_n)$$

Geometric multiplicity
of the eigenvalue λ_j



dimension of
the eigenspace \mathcal{E}_{λ_j}

Example #1: $A = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$

$$(A - 4I_2)u = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_2 \\ 0 \end{bmatrix}$$

$$(A - 4I_2)u = \mathbf{0} \Leftrightarrow \begin{bmatrix} u_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow u = \begin{bmatrix} u_1 \\ 0 \end{bmatrix} = u_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\lambda = 4$, algebraic multiplicity = 2

so, if $u \in \mathcal{E}_4 \Rightarrow u = \text{const.} \times \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Rightarrow \mathcal{E}_4 = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \Rightarrow \dim(\mathcal{E}_4) = 1$

\Rightarrow **geometric multiplicity = 1**

Example #2:

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

$$\begin{cases} \lambda_1 = 8 & \text{has algebraic multiplicity 1} \\ \lambda_2 = -1 & \text{has algebraic multiplicity 2} \end{cases}$$

$$\lambda_2 = -1$$

$$(A + I_3)u = 0 \Leftrightarrow \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow 2u_1 + u_2 + 2u_3 = 0$$
$$\Rightarrow u_2 = -2u_1 - 2u_3$$

$u_1 \in \mathbb{R}$
 $u_3 \in \mathbb{R}$
arbitrary

$$\text{so, if } u \in \mathcal{E}_{-1} \Rightarrow u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ -2u_1 - 2u_3 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ -2u_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2u_3 \\ u_3 \end{bmatrix}$$

$$\Rightarrow u = u_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + u_3 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \Rightarrow \mathcal{E}_{-1} = \text{span} \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\} \Rightarrow \dim(\mathcal{E}_{-1}) = 2$$

\Rightarrow geometric multiplicity of λ_2 is 2

$$\lambda_1 = 8$$

$$(A - 8I_3)u = 0 \Leftrightarrow \begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (*)$$

$$R_2 \rightarrow R_2 + 4R_1$$

$$R_3 \rightarrow R_3 - R_1$$

$$R_2 \rightarrow R_2/18$$

$$R_3 \rightarrow R_3 + 9R_2$$

$$R_1 \rightarrow R_1 - 5R_2$$

$$\left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 2 & -8 & 2 & 0 \\ 4 & 2 & -5 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ -18 & 0 & 18 & 0 \\ 9 & 0 & -9 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ -1 & 0 & 1 & 0 \\ 9 & 0 & -9 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 0 & 2 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$(*) \Leftrightarrow \begin{bmatrix} 0 & 2 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} u_3 = 2u_2 \\ u_1 = u_3 \end{cases}$$

$$\text{so, if } u \in \mathcal{E}_8 \Rightarrow u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 2u_2 \\ u_2 \\ 2u_2 \end{bmatrix} = u_2 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \mathcal{E}_8 = \text{span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\} \Rightarrow \dim(\mathcal{E}_8) = 1$$

\Rightarrow geometric multiplicity of λ_1 is 1

	Algebraic Multiplicity	Geometric Multiplicity
<div style="border: 1px dashed black; padding: 2px; display: inline-block;">Example #1</div> $\lambda_1 = 4$	2	1
<div style="border: 1px dashed black; padding: 2px; display: inline-block;">Example #2</div> $\lambda_1 = 8$ $\lambda_2 = -1$	1 2	1 2

Theorem:

Let λ be an eigenvalue of $\mathbf{A} \in \mathcal{M}_{n \times n}(\mathbb{R})$. Then

$1 \leq$ geometric multiplicity of $\lambda \leq$ algebraic multiplicity of λ