Differential Forms:
geometrical interpretation

Ciprian D. Coman

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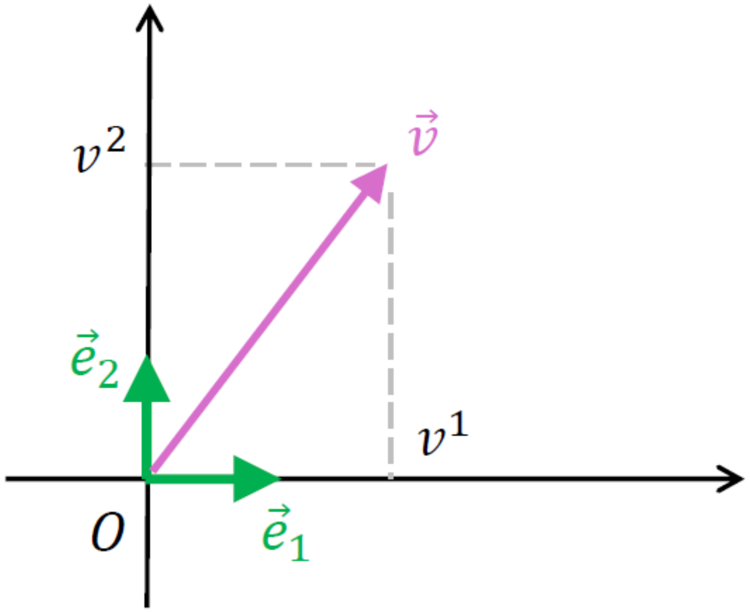

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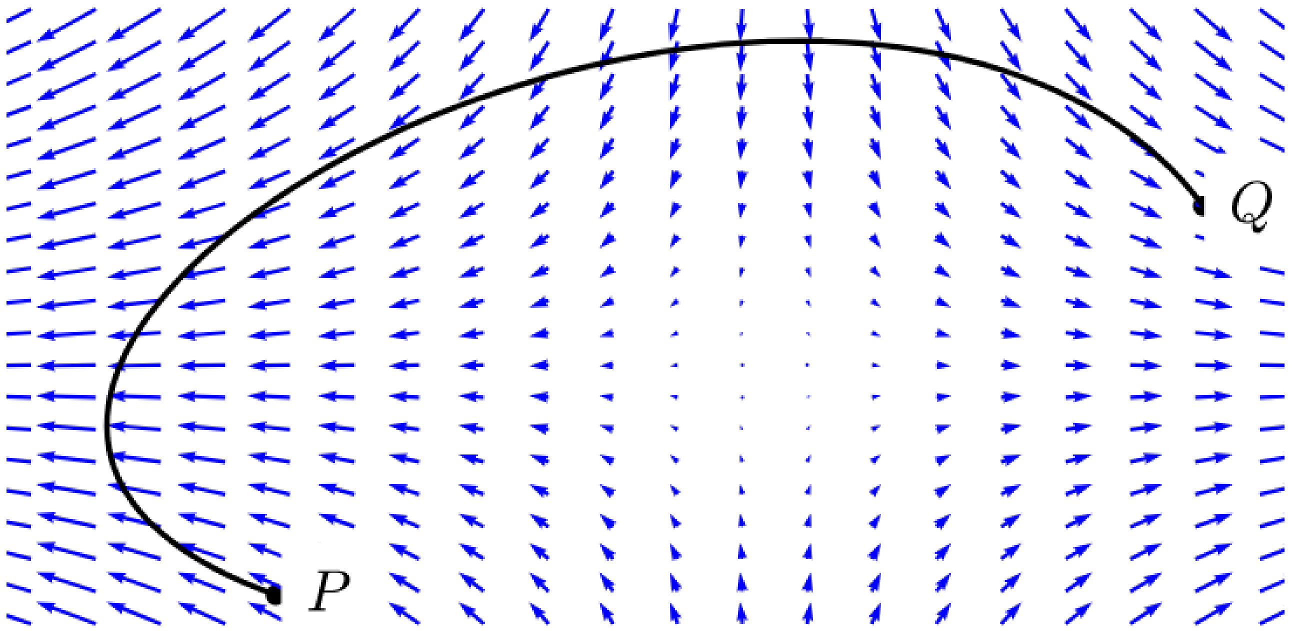
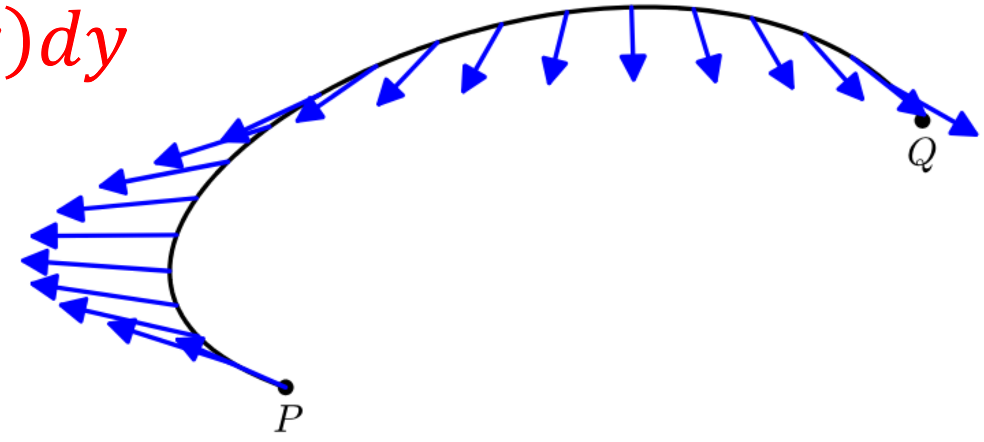
$$\vec{v} = v^1 \vec{e}_1 + v^2 \vec{e}_2 \qquad (v^1, v^2 \in \mathbb{R})$$

Familiar examples

$$\int_a^b f(x)dx$$



$$\int_{PQ} F(x,y)dx + G(x,y)dy$$



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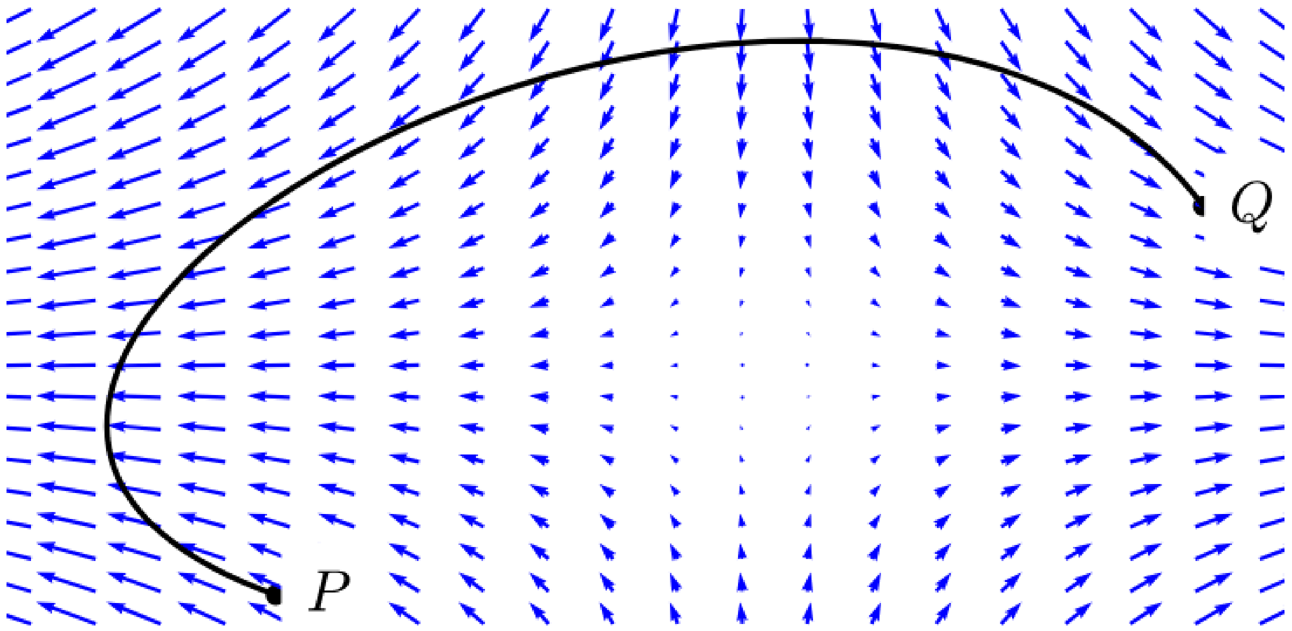
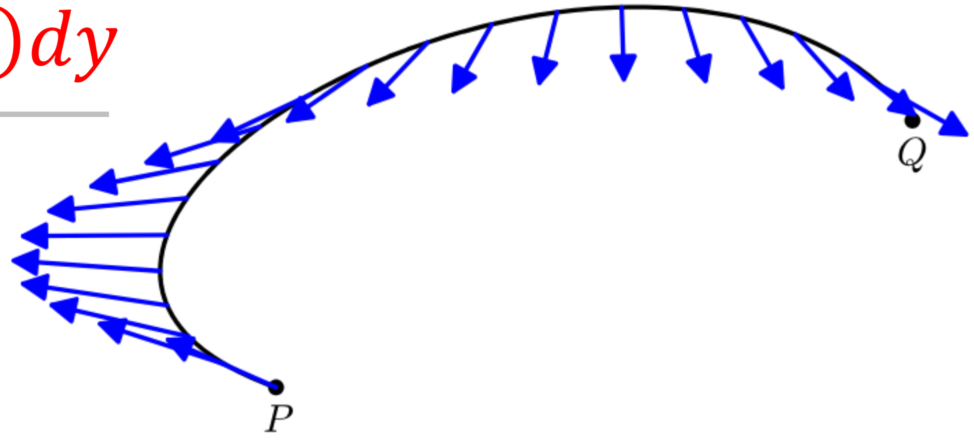


$$\int_a^b f(x)dx$$

"SIMPLEST"
DIFFERENTIAL 1-FORMS

"FIELD" OF **COVECTORS**

$$\int_{PQ} F(x,y)dx + G(x,y)dy$$

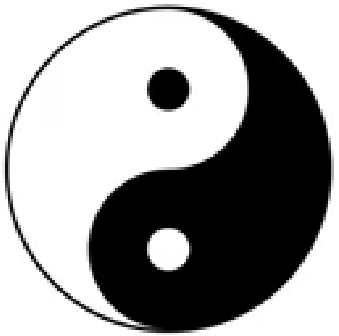


Vector-Covector Duality

Broadly speaking:

Vectors are objects that *"get measured"* (YIN)

Covectors are objects that *"measure"* (YANG)



Covectors are "basically" row vectors

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 \end{bmatrix}$$

column vector

row vector

Row vectors are functions on (column) vectors….

particular input (**vector**)

$$\begin{bmatrix} 2 & 1 \end{bmatrix}\left(\begin{bmatrix} 3 \\ 5 \end{bmatrix}\right) = ?$$

"the function"
(i.e., our **covector**)

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$$\begin{bmatrix} 2 & 1 \end{bmatrix}\left(\begin{bmatrix} 3 \\ 5 \end{bmatrix}\right) = (2)(3) + (1)(5) = 11$$

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(using the definition of matrix multiplication)

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…and so on

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COVECTOR: VECTOR → NUMBER

More formally...

... a **covector** on $V \equiv \mathbb{R}^2$ is a linear function

$$\alpha : V \to \mathbb{R}$$

Properties:

- ❏ $\alpha(\vec{v} + \vec{w}) = \alpha(\vec{v}) + \alpha(\vec{w})$
- ❏ $\alpha(m\vec{v}) = m\,\alpha(\vec{v})$

for all $\vec{v}, \vec{w} \in V$ and $m \in \mathbb{R}$

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The set of all such linear functions forms a vector space, $V^* =$ the dual of V

How do we visualise covectors (which are functions)?

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generic input (**vector**)

$$\begin{bmatrix} 2 & 1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 2x + 1y$$

our particular **covector**

How do we visualise covectors (which are functions)?

$$\begin{bmatrix} 2 & 1 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = 2x + 1y$$

LEVEL SETS:

$$2x + 1y = 0 \quad \rightarrow y = -2x$$

$$2x + 1y = 1 \quad \rightarrow y = -2x + 1$$

$$2x + 1y = 2 \quad \rightarrow y = -2x + 2$$

$$2x + 1y = -1 \quad \rightarrow y = -2x - 1$$

and so on....

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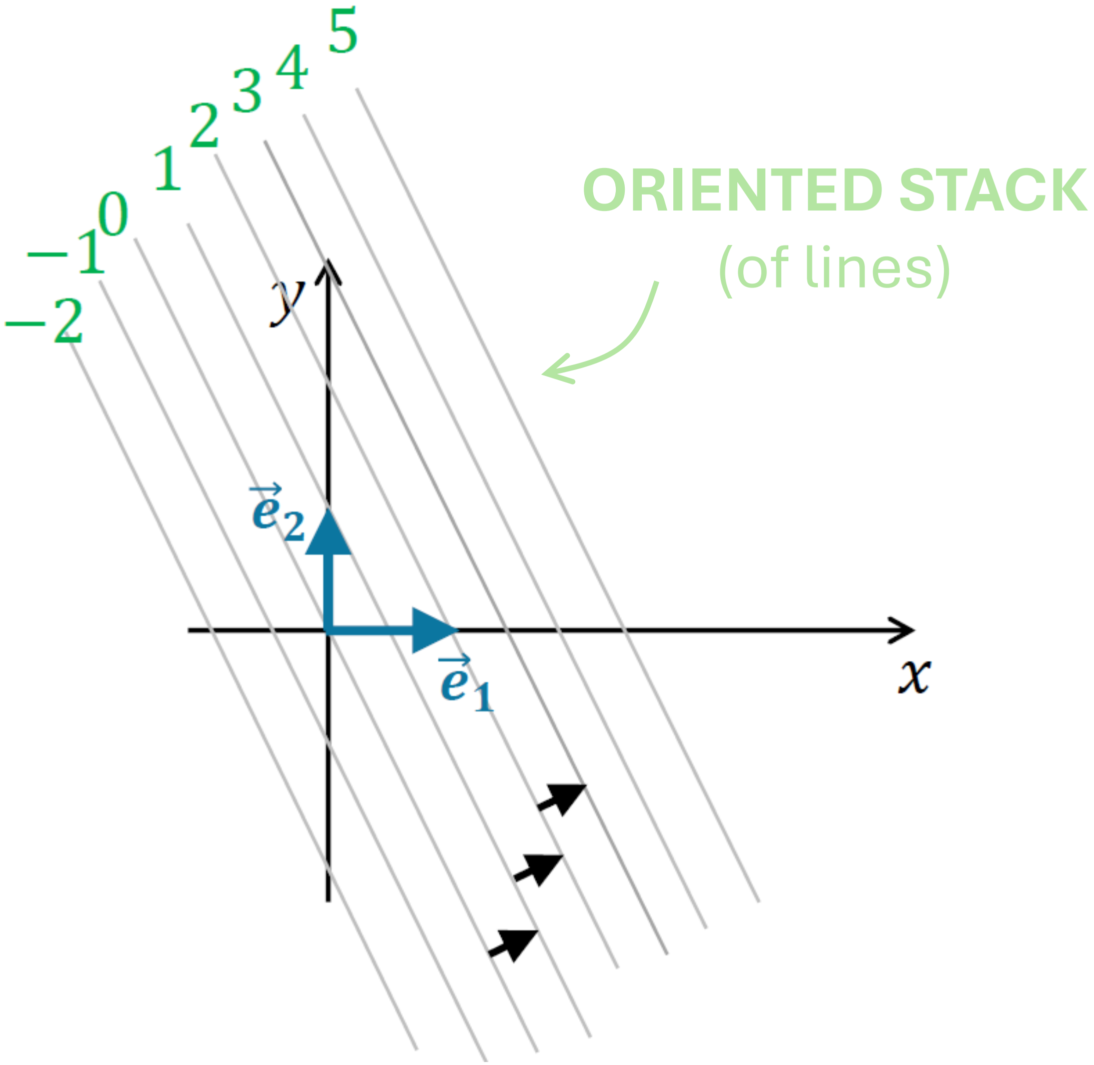

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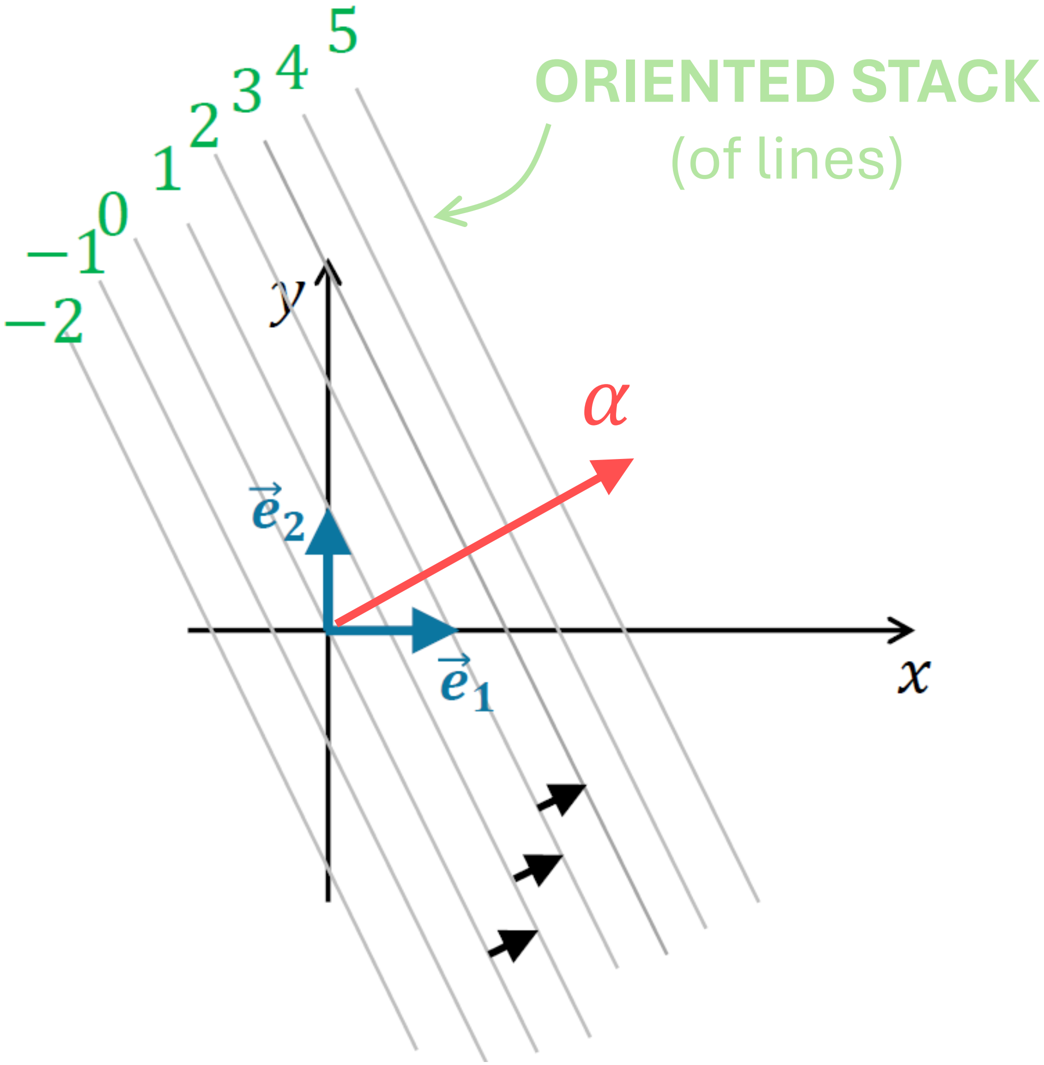
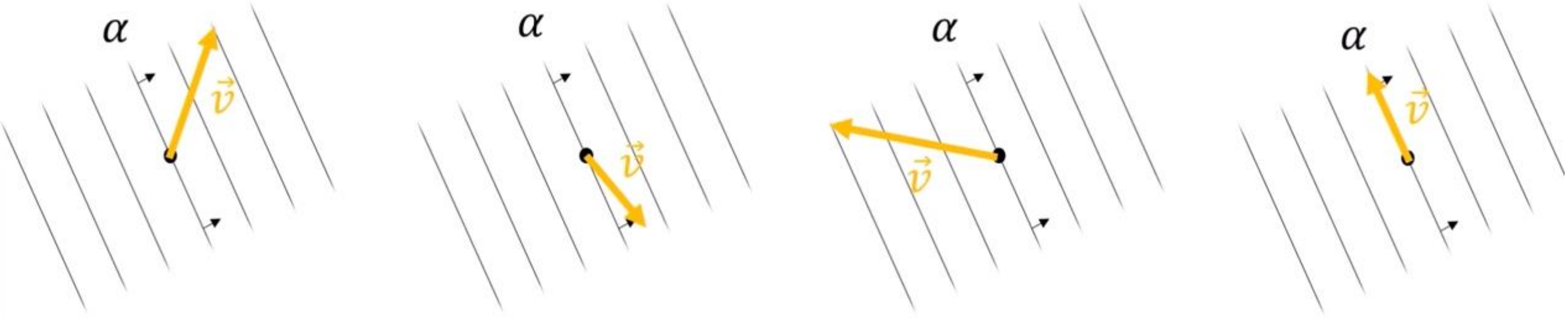


Intuitive calculations…

A covector α **measures** how directions (e.g., $\vec{v}$) cut across level sets



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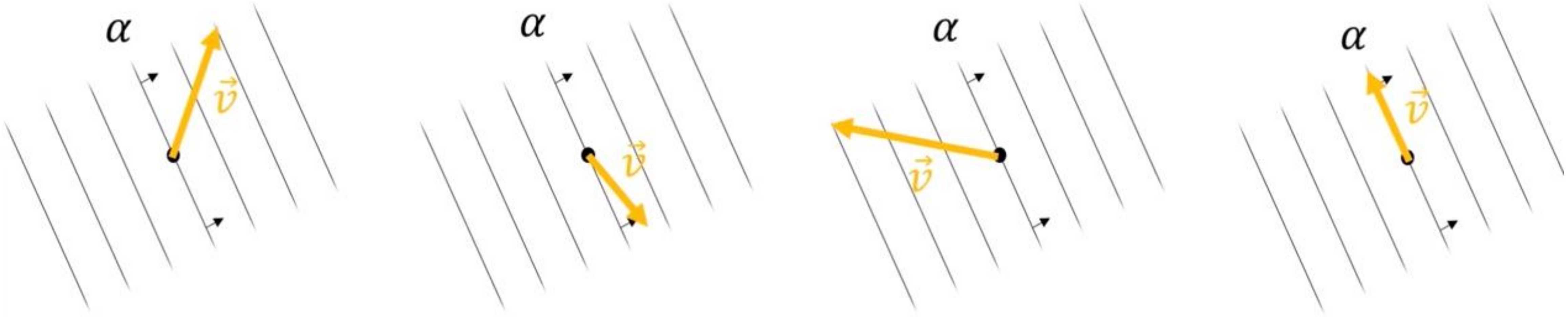


$\alpha(\vec{v}) = 2$ $\alpha(\vec{v}) = 0.5$ $\alpha(\vec{v}) = -3$ $\alpha(\vec{v}) = 0$

Re-interpretation: dx and dy → DIRECTIONAL DERIVATIVES

$$dx\left(\begin{bmatrix} v^1 \\ v^2 \end{bmatrix}\right) = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \end{bmatrix}}_{\text{GRADIENT OF } x} \left(\begin{bmatrix} v^1 \\ v^2 \end{bmatrix}\right) = [1 \quad 0]\left(\begin{bmatrix} v^1 \\ v^2 \end{bmatrix}\right) = v^1$$

$$dy\left(\begin{bmatrix} v^1 \\ v^2 \end{bmatrix}\right) = \underbrace{\begin{bmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \end{bmatrix}}_{\text{GRADIENT OF } y} \left(\begin{bmatrix} v^1 \\ v^2 \end{bmatrix}\right) = [0 \quad 1]\left(\begin{bmatrix} v^1 \\ v^2 \end{bmatrix}\right) = v^2$$

Re-interpret the d symbol....

OLD: Variable $\xrightarrow{d}$ "Small change" in Variable

x $\quad\longrightarrow\quad$ dx

Re-interpret the d symbol....

OLD: Variable x $\xrightarrow{d}$ "Small change" in Variable dx

NEW: **Scalar Field** f $\xrightarrow{d}$ **Covector Field** df

Directional Derivative

(1-differential form on $\mathbb{R}^2$)

Aside: **directional derivative** at a point P

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given scalar field

- $df_P : T_P(\mathbb{R}^2) \longrightarrow \mathbb{R}$ linear mapping
- can be expressed as

$$\boxed{df_P(\vec{v}) = \nabla f(P) \cdot \vec{v}} \qquad \text{for all } \vec{v} \in T_P(\mathbb{R}^2)$$

1-forms: general expression

$$\omega : \Omega \subseteq \mathbb{R}^2 \to \bigcup_{P \in \Omega} T_P(\mathbb{R}^2)$$

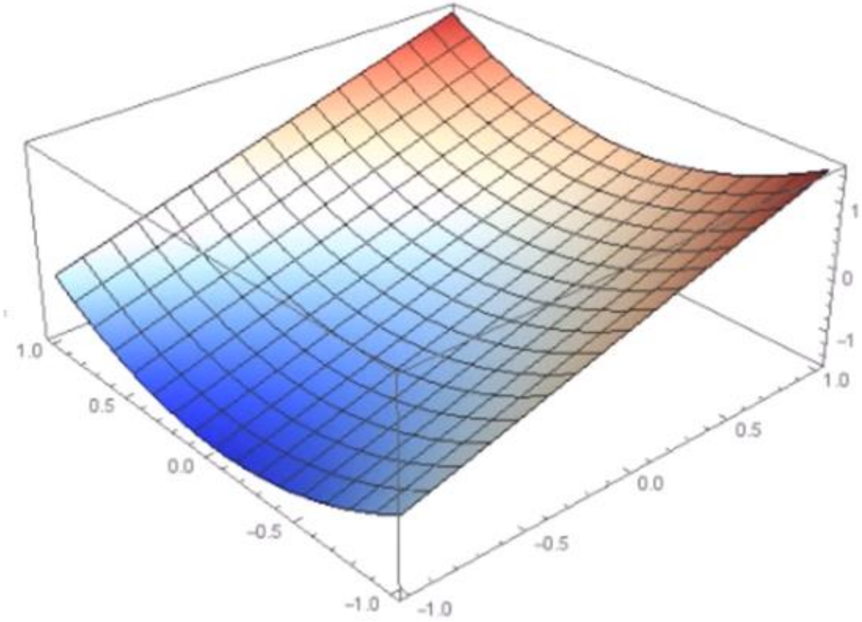
$$P \in \Omega \longrightarrow \omega_P \in (T_P(\mathbb{R}^2))^*$$

$$\omega_P = a(P)dx + b(P)dy$$ linear

given functions of **position**

(density plot)

(3D plot)



$$f(x, y) = y^2 + x - \frac{1}{2}$$

Scalar Field (e.g., temperature)

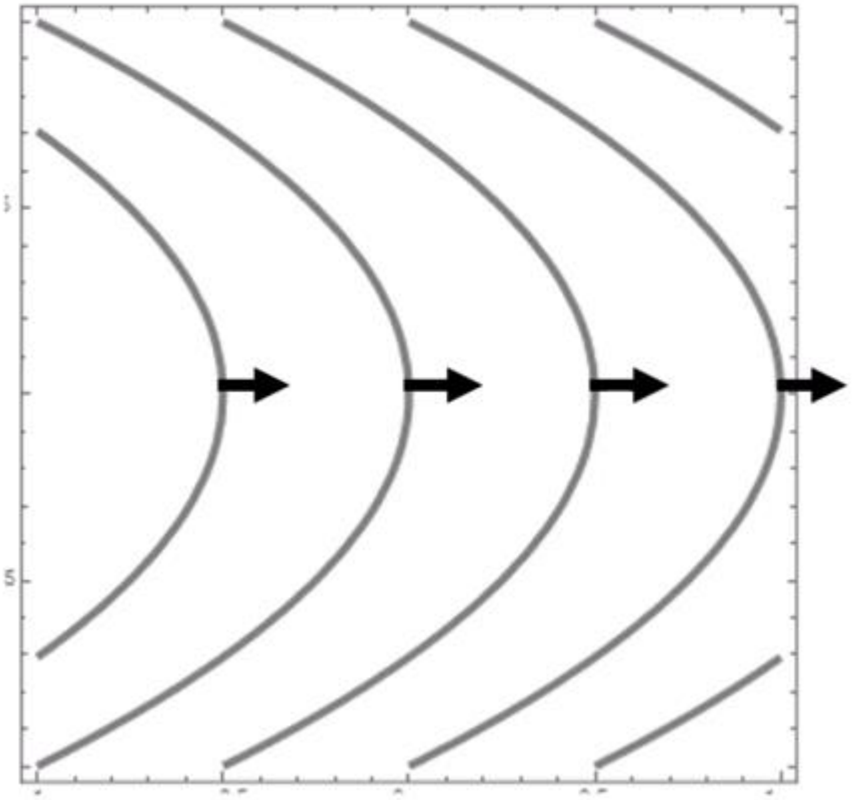
f

(density plot)

Scalar Field (e.g., temperature)

f

d



Scalar Field (e.g., temperature)

f

Covector Field

df

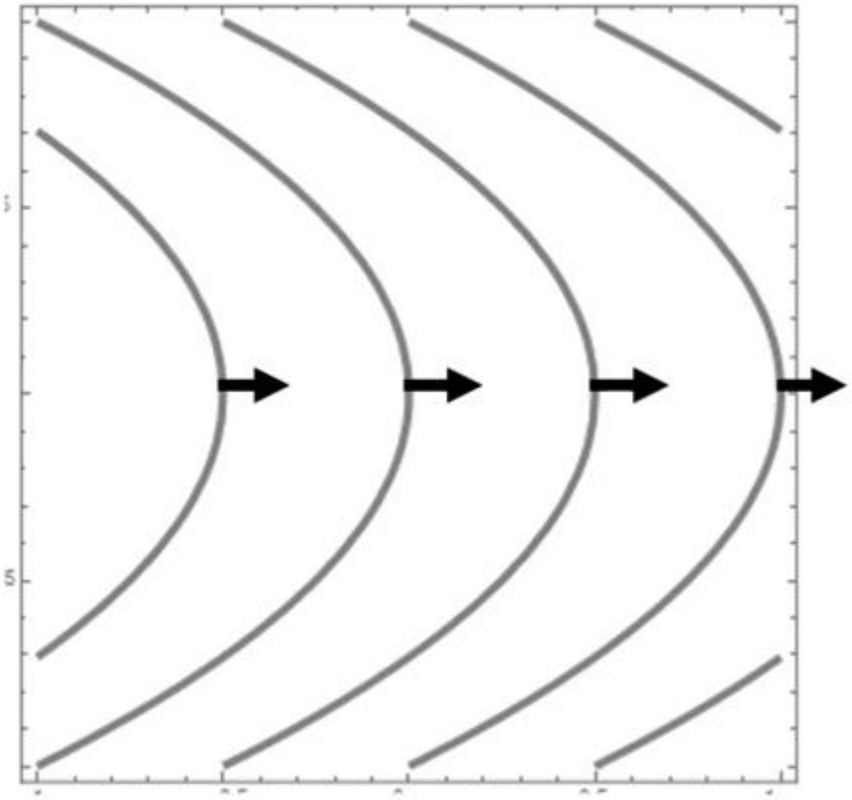
(1-differential form on $\mathbb{R}^2$)

Scalar Field (e.g., temperature)

f

d

$\cdots f = -1 \quad f = 0 \quad f = 1 \quad f = 2 \cdots$

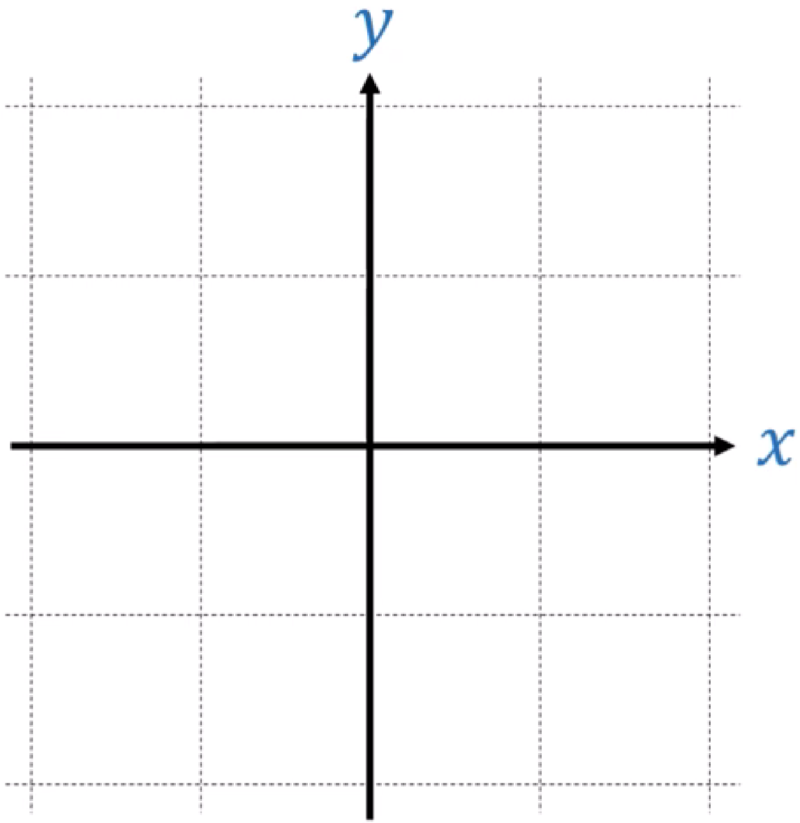


Covector Field

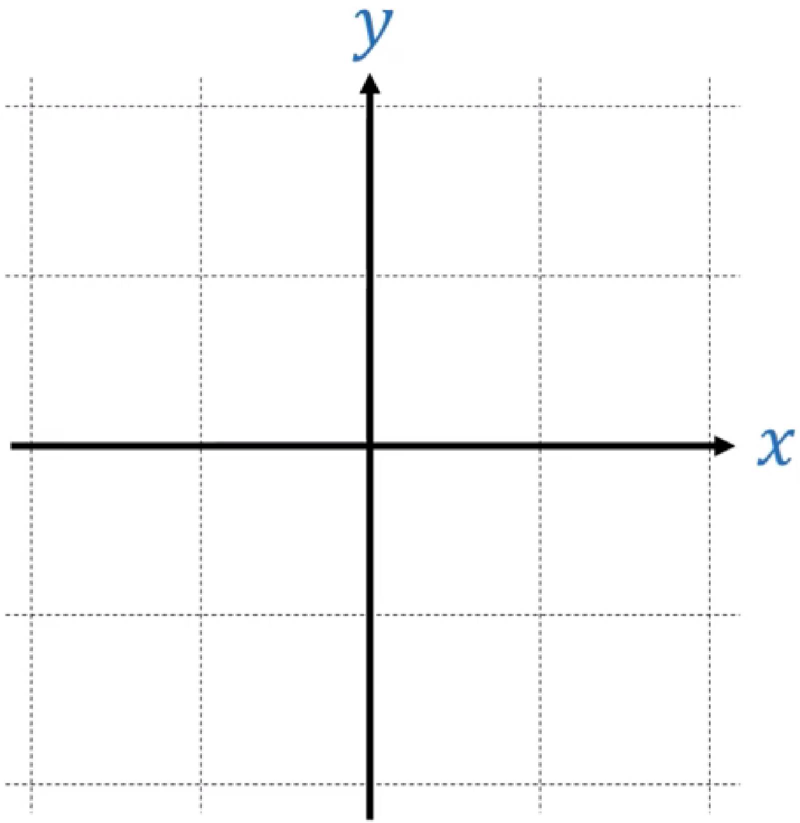
df

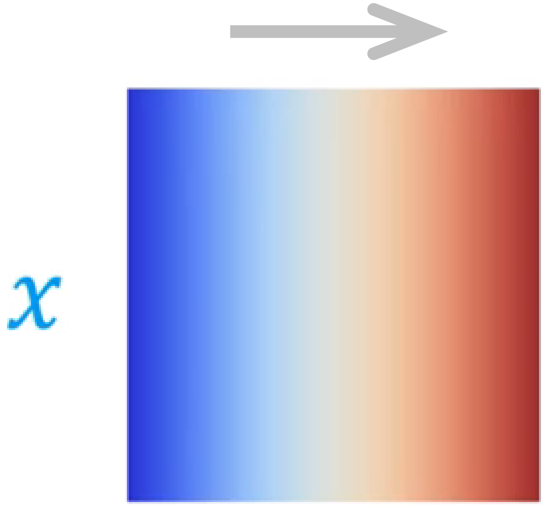
(1-differential form on $\mathbb{R}^2$)

dx and dy



dx and dy





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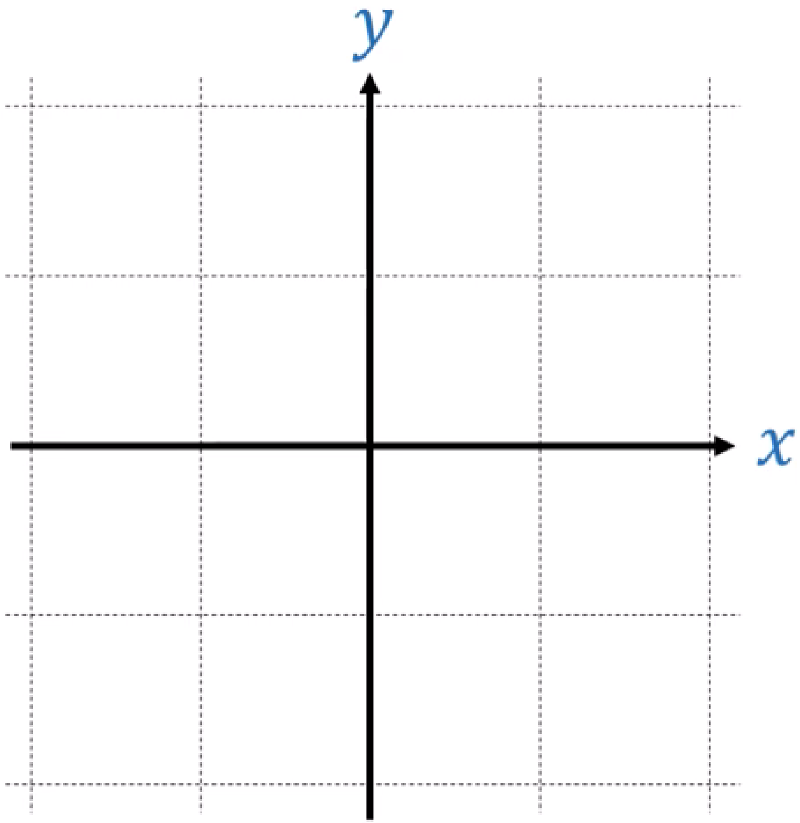

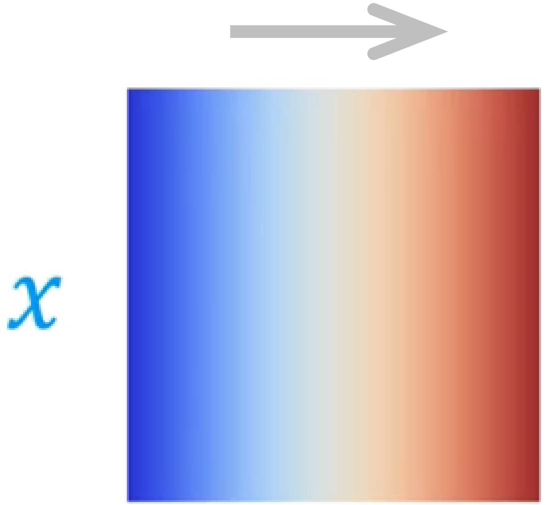

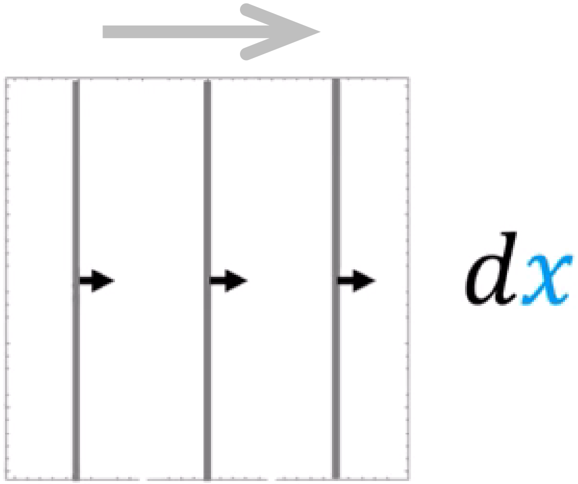

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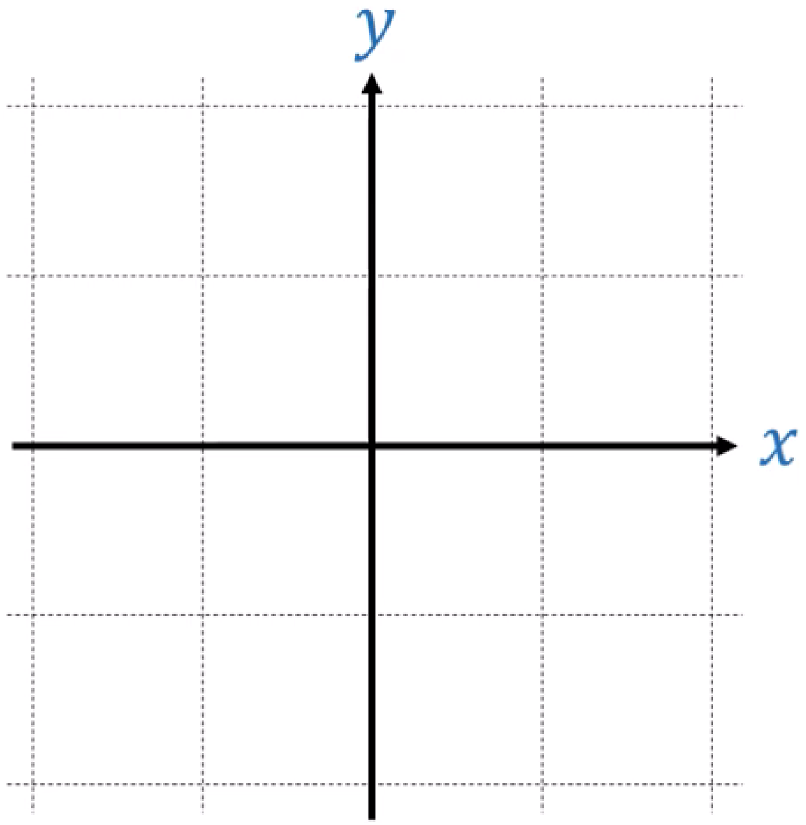

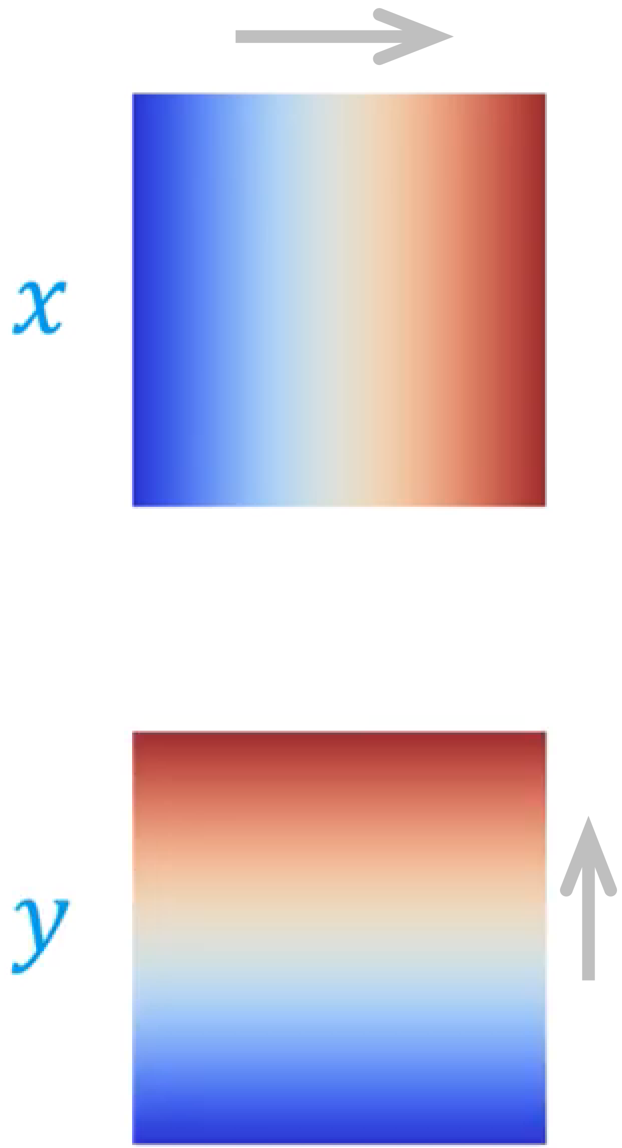

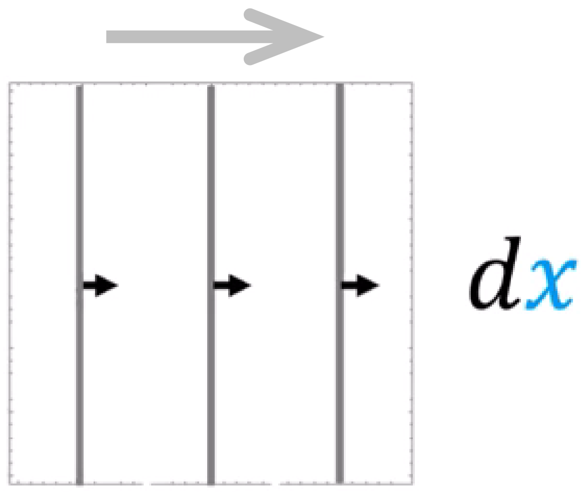

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