An introduction to determinants

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Preliminaries:
(notation)

(3 × 3 matrix)

$$\begin{array}{cc} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\ & \begin{matrix} 1 & \quad 2 & \quad 3 \end{matrix} \end{array}$$

$$\begin{array}{cc} & \begin{matrix} 1 & \quad 2 & \quad 3 & \quad 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \end{array}$$

(4 × 4 matrix)

general notation

$$\boxed{a_{ij}}$$

i: row #

j: column #

Each **square** matrix A has a **determinant**, a real number denoted either by $\det(A)$ or $|A|$

2 x 2 matrices:

The **determinant** of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is given by

$$\det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}.$$

An easy way for remembering:

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Examples: Find the determinant of each matrix:

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix}$$

$$= 2(2) - 1(-3)$$

$$= 4 + 3 = 7$$

$$|B| = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix}$$

$$= 2(2) - 4(1)$$

$$= 4 - 4 = 0$$

$$|C| = \begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix}$$

$$= 0(4) - 2(3)$$

$$= 0 - 6 = -6$$

Definition (minors & cofactors):

If A is an $(n \times n)$ matrix, then the **minor** M_{ij} of the element a_{ij} is the **determinant** of the matrix obtained by deleting the *i*-th row and *j*-th column of *A*. The **cofactor** C_{ij} is defined as

$$\boxed{C_{ij} = (-1)^{i+j} M_{ij}} = \begin{cases} +M_{ij}\,, & \text{if} \quad i+j \text{ is even} \\ -M_{ij}\,, & \text{if} \quad i+j \text{ is odd} \end{cases}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow \begin{array}{cc} & \begin{matrix} \phantom{a_{11}} & \text{delete} & \phantom{a_{13}} \end{matrix} \\ \begin{matrix} \\ \\ \text{delete} \end{matrix} & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array} \rightarrow \underbrace{\begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}}_{2 \times 2 \text{ sub-matrix}}$$

Minor of a_{21}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

Delete **row 2** and **column 1**

Cofactor of a_{21}

$$C_{21} = (-1)^{2+1} M_{21}$$
$$= -M_{21}$$

Minor of a_{22}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

Delete **row 2** and **column 2**

Cofactor of a_{22}

$$C_{22} = (-1)^{2+2} M_{22}$$
$$= M_{22}$$

The general case:

Determinants are defined *inductively*: those of matrices of size $n \times n$ are defined in terms of determinants of matrices of size $(n-1) \times (n-1)$ for $n = 2, 3, \ldots$. More precisely,

$n \times n$ determinant $=$ **linear combination** of $(n-1) \times (n-1)$ determinants

$n = 1$ $\qquad$ $n = 2$ $\qquad$ $n = 3$ $\qquad$ $n = 4$ $\qquad$ …… etc

$$A = [a_{11}]$$

we define:

$$|A| = a_{11}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

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$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Definition ("expansion by cofactors in the first row"):

If A is an $(n \times n)$ **square** matrix $(n \geq 2)$, then the determinant of A is the sum of the **entries** in the first row of A *multiplied* by **their cofactors**:

$$\det(A) = |A| = \underbrace{a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}}_{\sum_{j=1}^{n} a_{1j}C_{1j}}$$

$n = 2$: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\qquad \det(A) \equiv |A| = a_{11}C_{11} + a_{12}C_{12} = a_{11}a_{22} + a_{12}(-a_{21})$

$$= a_{11}a_{22} - a_{12}a_{21}$$

REMINDER

$$C_{ij} = (-1)^{i+j} M_{ij}$$

cofactor minor

Examples: 1). $A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix} \rightarrow M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1(1) - 0(2) = -1 \qquad C_{11} = -1$$

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix} \rightarrow M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 3(1) - 4(2) = -5 \qquad C_{12} = 5$$

$$\ldots M_{13} = 4 \qquad C_{13} = 4$$

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= 0(-1) + 2(5) + 1(4) = 14$$

2).

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

REMINDER

$$\underbrace{C_{ij}}_{\text{cofactor}} = (-1)^{i+j} \underbrace{M_{ij}}_{\text{minor}}$$

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

(first-row expansion formula)

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = 5 \times 9 - 6 \times 8 = 45 - 48 = -3$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = -(4 \times 9 - 6 \times 7) = -(36 - 42) = -(-6) = 6$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 4 \times 8 - 5 \times 7 = 32 - 35 = -3$$

$$\det(A) = 1(-3) + 2(6) + 3(-3) = 0$$