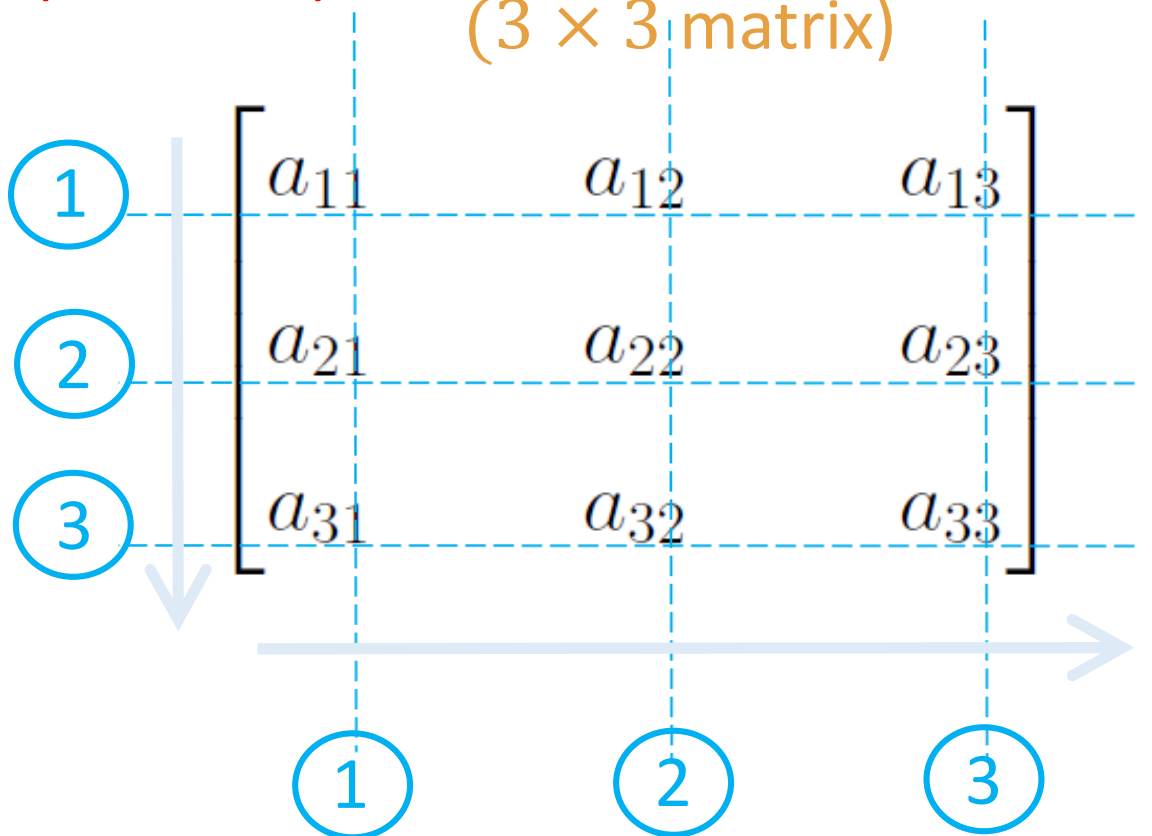


# **An introduction to determinants**

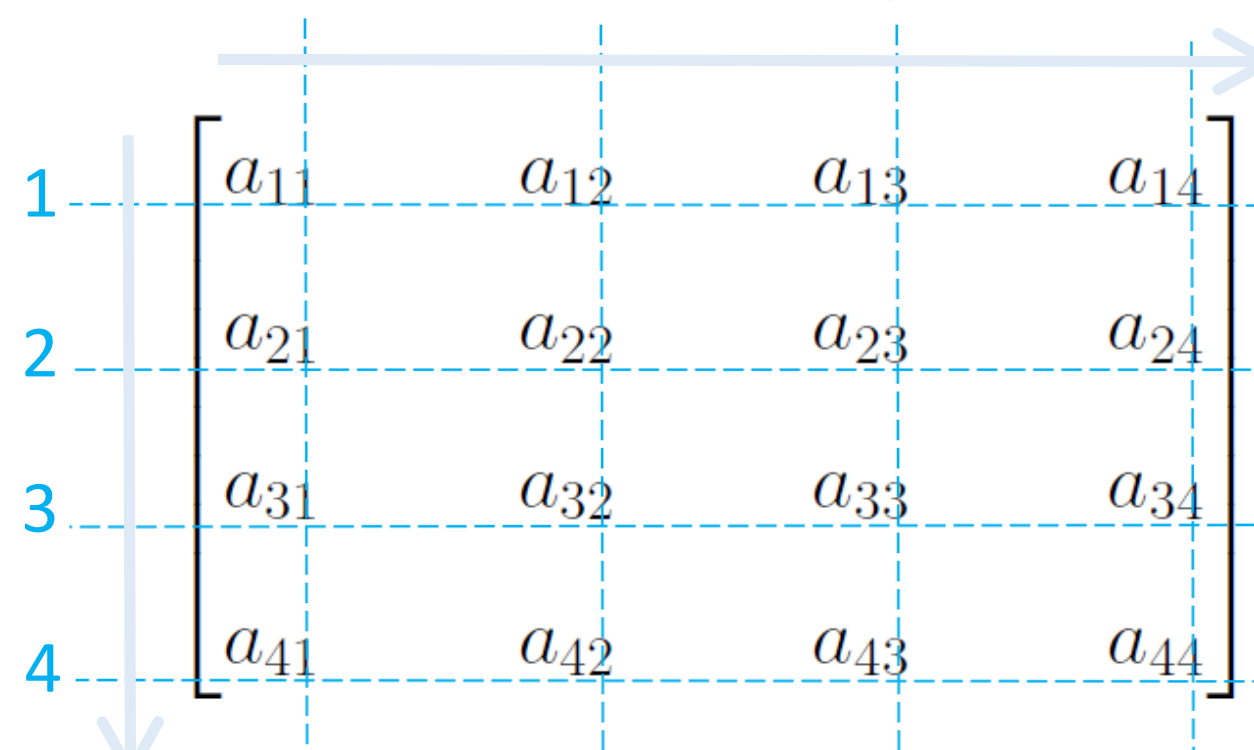
C. D. Coman

**Preliminaries:**  
(notation)

(3 × 3 matrix)

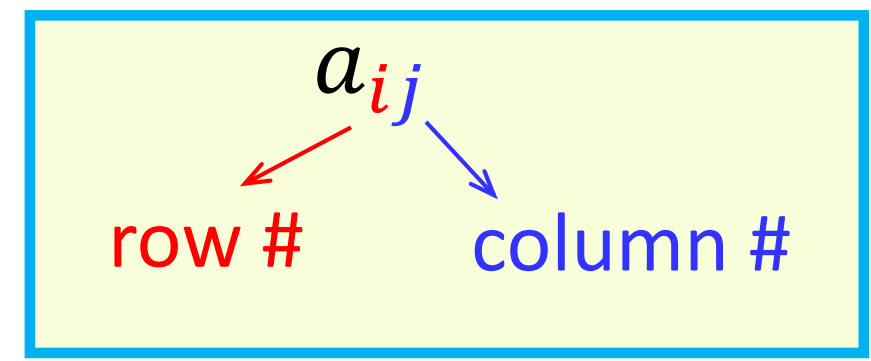
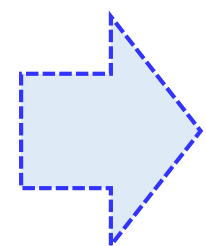


1 2 3 4



(4 × 4 matrix)

general notation



Each **square** matrix  $A$  has a **determinant**, a real number denoted either by  $\det(A)$  or  $|A|$

**2 x 2 matrices:**

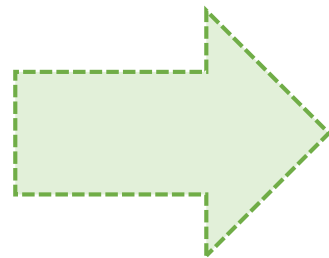
The **determinant** of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is given by

$$\det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}$$

An easy way for remembering:



$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

The diagram shows a 2x2 determinant with blue arrows indicating the calculation: a diagonal arrow from  $a_{11}$  to  $a_{22}$  and another diagonal arrow from  $a_{21}$  to  $a_{12}$ . The result is the difference of the products of these two diagonals.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

**Examples:** Find the determinant of each matrix:

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\begin{aligned} |A| &= \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} \\ &= 2(2) - 1(-3) \\ &= 4 + 3 = 7 \end{aligned}$$

$$\begin{aligned} |B| &= \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} \\ &= 2(2) - 4(1) \\ &= 4 - 4 = 0 \end{aligned}$$

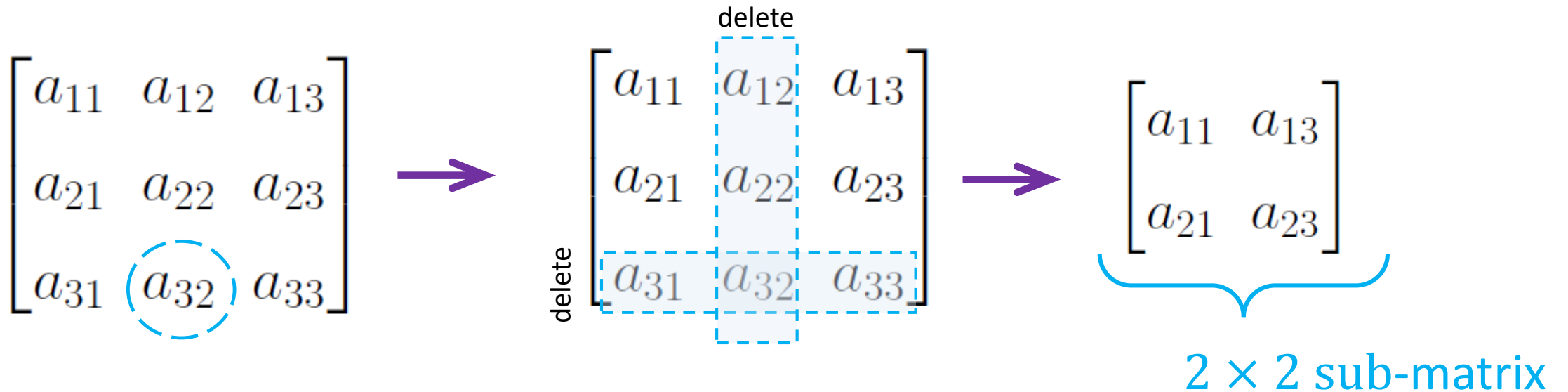
$$\begin{aligned} |C| &= \begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix} \\ &= 0(4) - 2(3) \\ &= 0 - 6 = -6 \end{aligned}$$

## Definition (minors & cofactors):

If  $A$  is an  $(n \times n)$  matrix, then the **minor**  $M_{ij}$  of the element  $a_{ij}$  is the **determinant** of the matrix obtained by deleting the  $i$ -th row and  $j$ -th column of  $A$ .

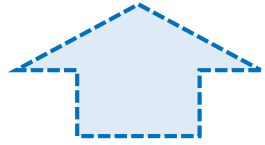
The **cofactor**  $C_{ij}$  is defined as

$$C_{ij} = (-1)^{i+j} M_{ij} = \begin{cases} +M_{ij}, & \text{if } i + j \text{ is even} \\ -M_{ij}, & \text{if } i + j \text{ is odd} \end{cases}$$



**Minor of  $a_{21}$**

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$



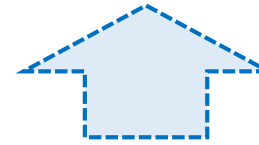
Delete row 2 and column 1

**Cofactor of  $a_{21}$**

$$\begin{aligned} C_{21} &= (-1)^{2+1} M_{21} \\ &= -M_{21} \end{aligned}$$

**Minor of  $a_{22}$**

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$



Delete row 2 and column 2

**Cofactor of  $a_{22}$**

$$\begin{aligned} C_{22} &= (-1)^{2+2} M_{22} \\ &= M_{22} \end{aligned}$$

# The general case:

Determinants are defined *inductively*: those of matrices of size  $n \times n$  are defined in terms of determinants of matrices of size  $(n - 1) \times (n - 1)$  for  $n = 2, 3, \dots$ . More precisely,

$n \times n$   
determinant

=

linear combination of  $(n - 1) \times (n - 1)$   
determinants

$n = 1$

$n = 2$

$n = 3$

$n = 4$

... etc

$$A = [a_{11}]$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$


we define:

$$|A| = a_{11}$$

**Definition** (“expansion by cofactors in the first row”):

If  $A$  is an  $(n \times n)$  **square** matrix ( $n \geq 2$ ), then the determinant of  $A$  is the sum of the **entries** in the first row of  $A$  *multiplied* by **their cofactors**:

$$\det(A) = |A| = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$


$$\sum_{j=1}^n a_{1j}C_{1j}$$

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$n = 2$ :

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\begin{aligned} \det(A) \equiv |A| &= a_{11}C_{11} + a_{12}C_{12} = a_{11}a_{22} + a_{12}(-a_{21}) \\ &= a_{11}a_{22} - a_{12}a_{21} \end{aligned}$$



Examples: 1).

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

REMINDER

$$C_{ij} = (-1)^{i+j} M_{ij}$$

↓ cofactor      ↓ minor

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix} \rightarrow M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1(1) - 0(2) = -1$$

$$C_{11} = -1$$

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix} \rightarrow M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 3(1) - 4(2) = -5$$

$$C_{12} = 5$$

$$\dots M_{13} = 4$$

$$C_{13} = 4$$

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= 0(-1) + 2(5) + 1(4) = 14$$

2).

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

REMINDER

$$C_{ij} = (-1)^{i+j} M_{ij}$$

↓ cofactor      ↓ minor

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \quad (\text{first-row expansion formula})$$

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = 5 \times 9 - 6 \times 8 = 45 - 48 = -3$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = -(4 \times 9 - 6 \times 7) = -(36 - 42) = -(-6) = 6$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 4 \times 8 - 5 \times 7 = 32 - 35 = -3$$

$$\det(A) = 1(-3) + 2(6) + 3(-3) = 0$$