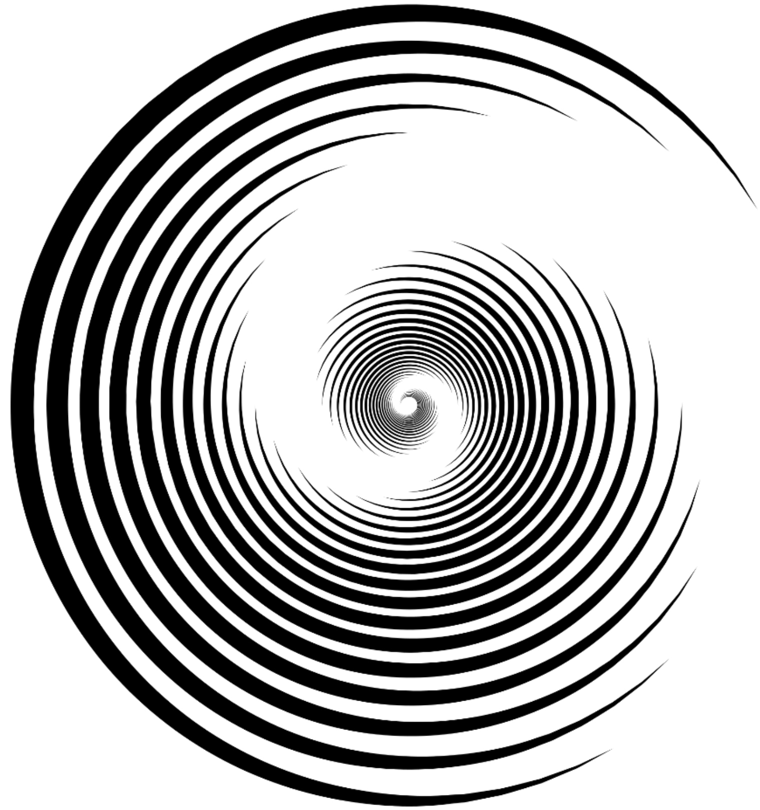
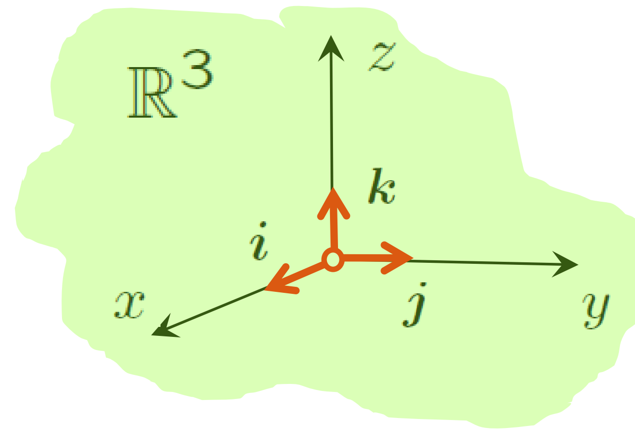
The ‘**curl**’ of a vector field

EEA_5_007



A first definition:



Let $\boldsymbol{F} = \boldsymbol{F}(x, y, z)$ be a vector field:

$$\boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k} \equiv (F_1, F_2, F_3)$$

The 'curl' of the above vector field is defined to be another vector field, denoted by $\mathrm{curl}\boldsymbol{F}$,and defined as:

$$\mathrm{curl}\boldsymbol{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k}$$

Questions:

- How can we remember this formula?
- Origin of its name?

The formula for the 'curl' is easier to remember if we rewrite it using "operator" notation. To this end, we introduce the symbol "**del**":

$$\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}$$

This acts, or "operates", on functions $f : \mathbb{R}^3 \to \mathbb{R}$ according to the rule:

$$f \longrightarrow \underline{\nabla f} = \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right) f = \underline{\boldsymbol{i}\frac{\partial f}{\partial x} + \boldsymbol{j}\frac{\partial f}{\partial y} + \boldsymbol{k}\frac{\partial f}{\partial z}}$$

the gradient of f

Important: We can view ∇ as a vector with components:

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$$

ASIDE:
(vector products)

$$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} \equiv (a_1, a_2, a_3)$$

$$\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k} \equiv (b_1, b_2, b_3)$$

$\boldsymbol{a} \times \boldsymbol{b}$

magnitude: $\|\boldsymbol{a} \times \boldsymbol{b}\| = \|\boldsymbol{a}\|\|\boldsymbol{b}\| \sin\theta$

$$\theta = \angle(\boldsymbol{a}, \boldsymbol{b})$$

direction = perpendicular on both vectors

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\boldsymbol{i} + (a_3b_1 - a_1b_3)\boldsymbol{j} + (a_1b_2 - a_2b_1)\boldsymbol{k}$$

Back to the 'curl':

$$\nabla \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \boldsymbol{i}(-1)^{1+1}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_2 & F_3 \end{vmatrix} + \boldsymbol{j}(-1)^{1+2}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ F_1 & F_3 \end{vmatrix} + \boldsymbol{k}(-1)^{1+3}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 \end{vmatrix}$$

Aside:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$a \to \frac{\partial}{\partial y} \quad b \to \frac{\partial}{\partial z}$$

$$c \to F_2 \quad d \to F_3$$

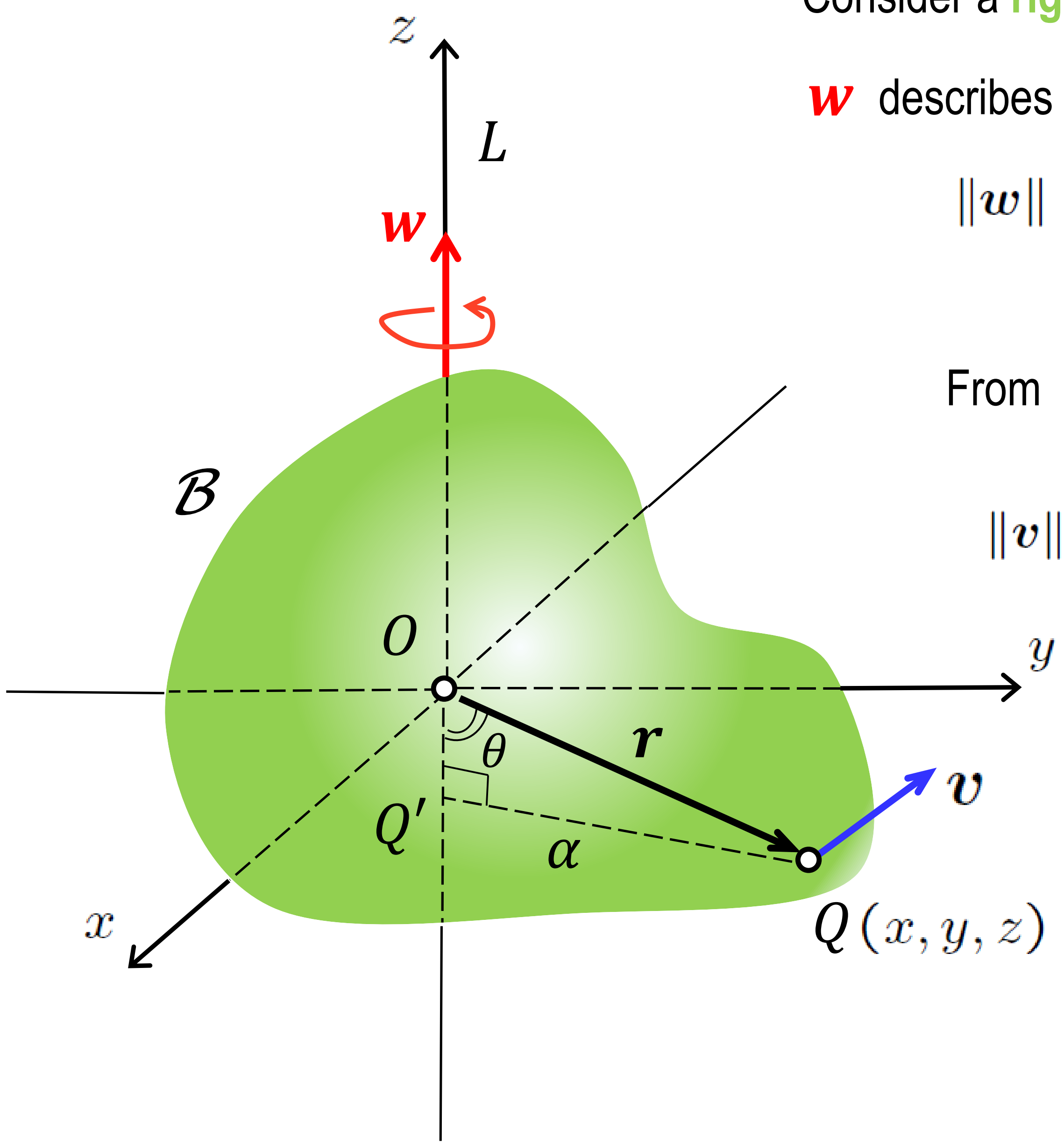
$$\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}$$

Etc.

$$\ldots\ldots = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k} = \text{curl}\boldsymbol{F}$$

A simpler expression for the 'curl':

$$\text{curl}\,\boldsymbol{F} = \nabla \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$



Consider a rigid body $\mathcal{B}$ rotating about an axis *L*.

$\boldsymbol{w}$ describes the rotational motion of the body:

$\|\boldsymbol{w}\| = \omega$ and $\boldsymbol{w} = \omega \boldsymbol{k}$ (1)

(ω: angular speed)

From $\Delta OQQ'$: $\alpha = \|\boldsymbol{r}\| \sin\theta$

$$\|\boldsymbol{v}\| = \omega\alpha = \omega\|\boldsymbol{r}\|\sin\theta = \underbrace{\|\boldsymbol{w}\|\|\boldsymbol{r}\|\sin\theta}_{\|\boldsymbol{w}\times\boldsymbol{r}\|}$$

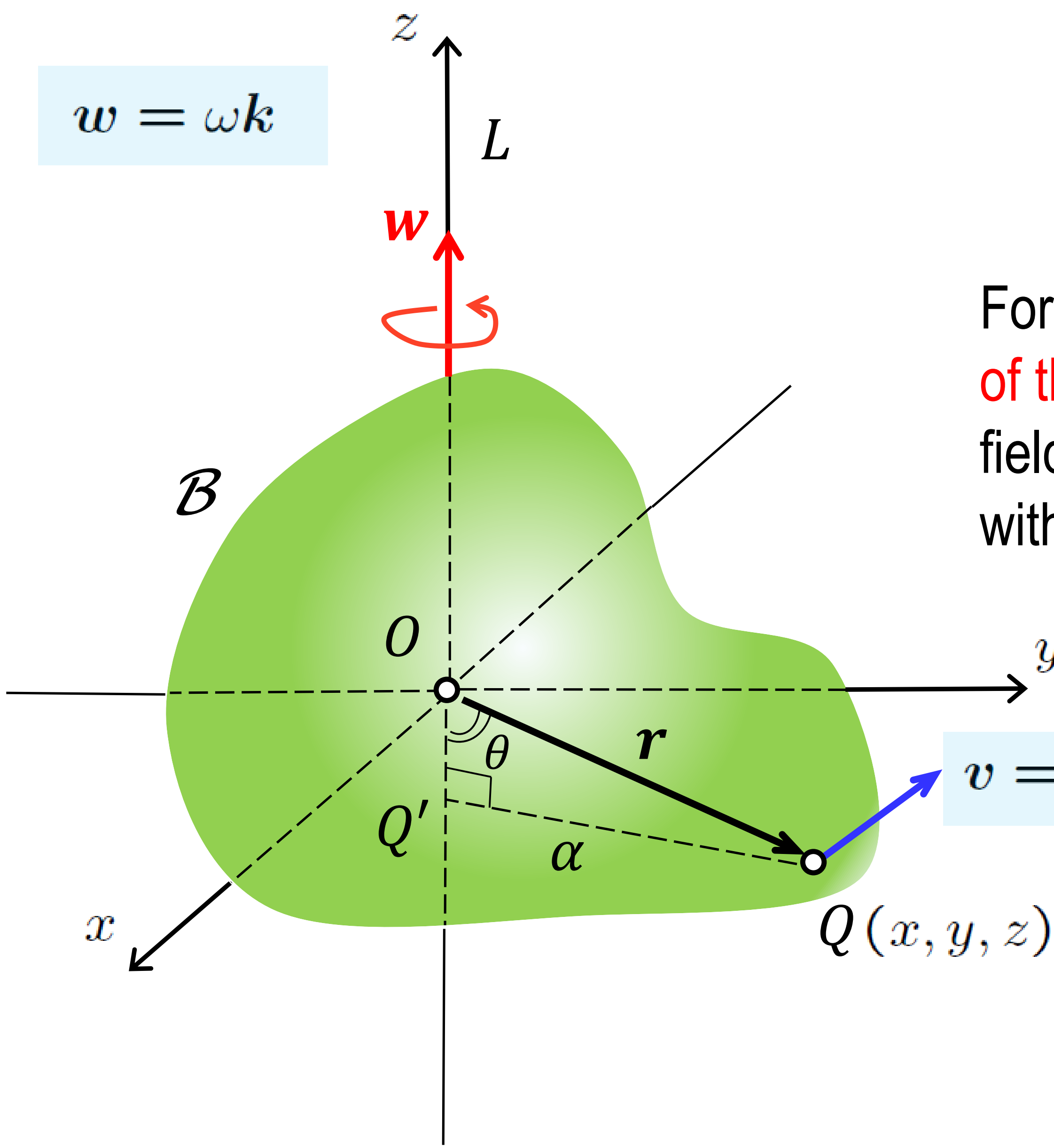
Hence $\boldsymbol{v} = \boldsymbol{w} \times \boldsymbol{r}$

(1) & (2) $\Rightarrow$ $\boldsymbol{v} = -\omega y \boldsymbol{i} + \omega x \boldsymbol{j}$

$$\operatorname{curl} \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega\boldsymbol{k} \overset{(1)}{=} 2\boldsymbol{w}$$

$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ (2)

$$\boldsymbol{w} = \omega \boldsymbol{k}$$



$$\boldsymbol{v} = \boldsymbol{w} \times \boldsymbol{r}$$

$$\nabla \times \boldsymbol{v} = 2\boldsymbol{w}$$

For the rotation of a rigid body the ‘curl’ of the velocity vector field is a vector field directed along the axis of rotation with magnitude twice the **angular speed**

shows why the curl is associated with rotations

Some definitions:

Let $\boldsymbol{F} = \boldsymbol{F}(x, y, z)$ be a vector field:

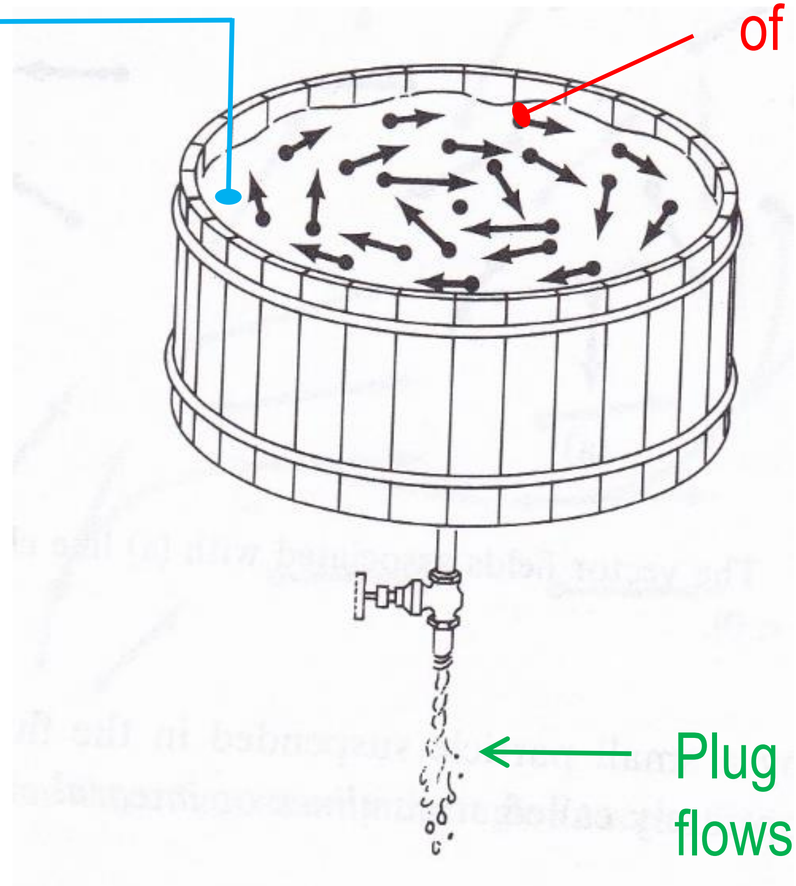
$$\boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k} \equiv (F_1, F_2, F_3)$$

$\nabla \times \boldsymbol{F} = 0$ → such vector fields $\boldsymbol{F}$ are called **irrotational**

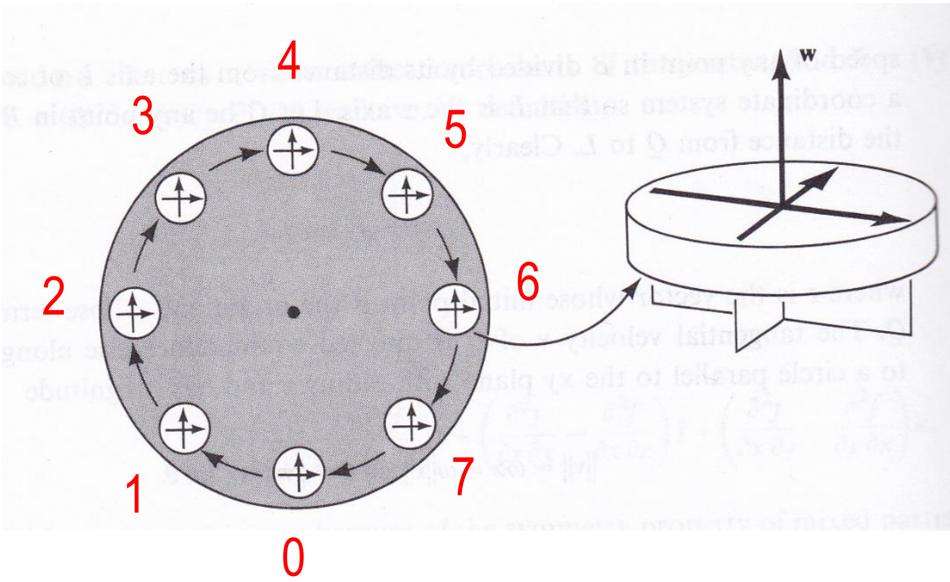
$\nabla \times \boldsymbol{F} \neq 0$ → .... $\boldsymbol{F}$ is called **rotational**

Liquid

The arrows indicate the velocity of various points on the surface of the fluid



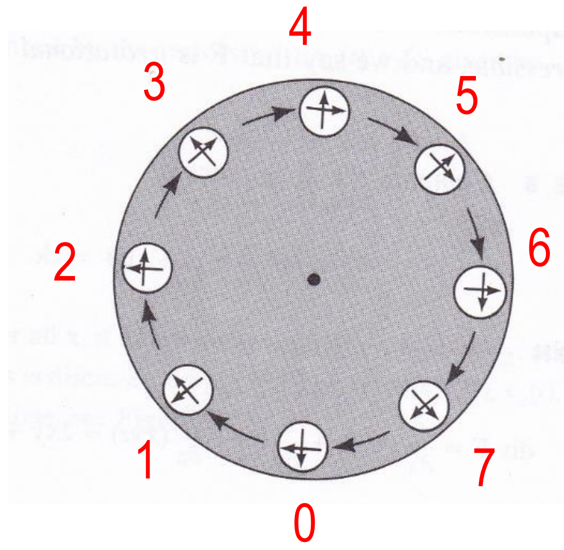
Plug is open (so that the fluid flows down)



$$V(x,y,z) = \frac{y\boldsymbol{i}}{x^2+y^2} - \frac{x\boldsymbol{j}}{x^2+y^2}$$

This velocity field is irrotational: a small paddle wheel moving in the fluid will not rotate around its axis ***w***

$$\nabla \times V = 0$$



$$V(x,y,z) = y\boldsymbol{i} - x\boldsymbol{j}$$

This velocity field is rotational: a small paddle wheel moving in the fluid rotates around its axis ***w***

$$\nabla \times V = -2\boldsymbol{k} \neq 0$$