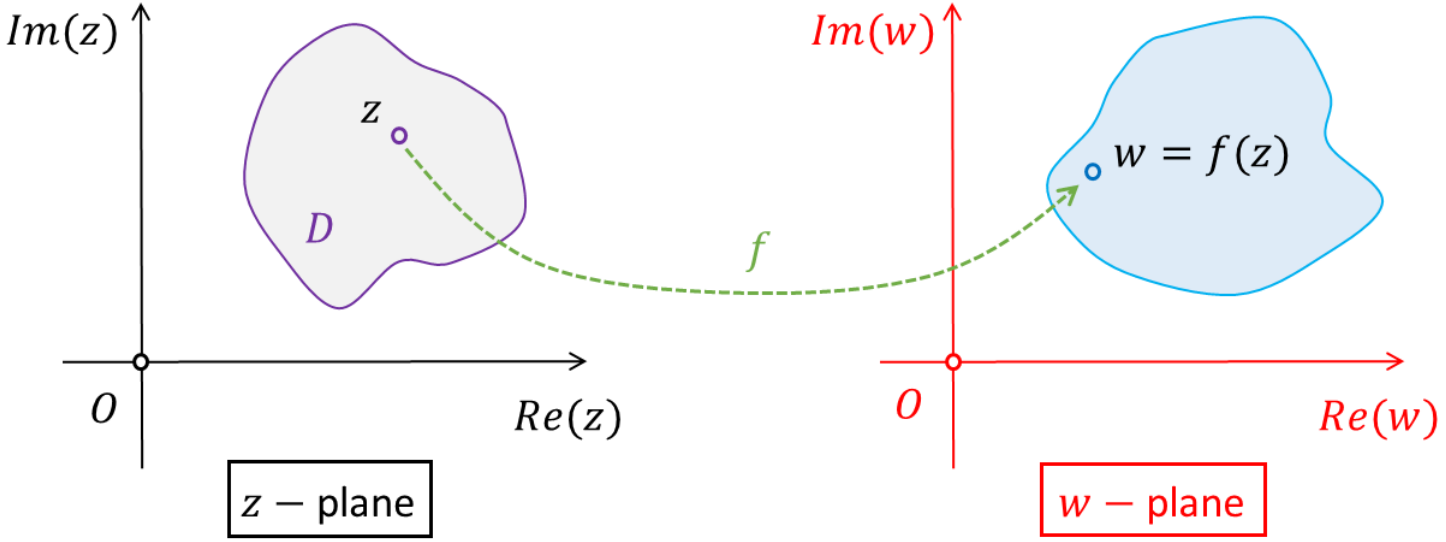
Complex calculus: differentiation

Preliminaries:

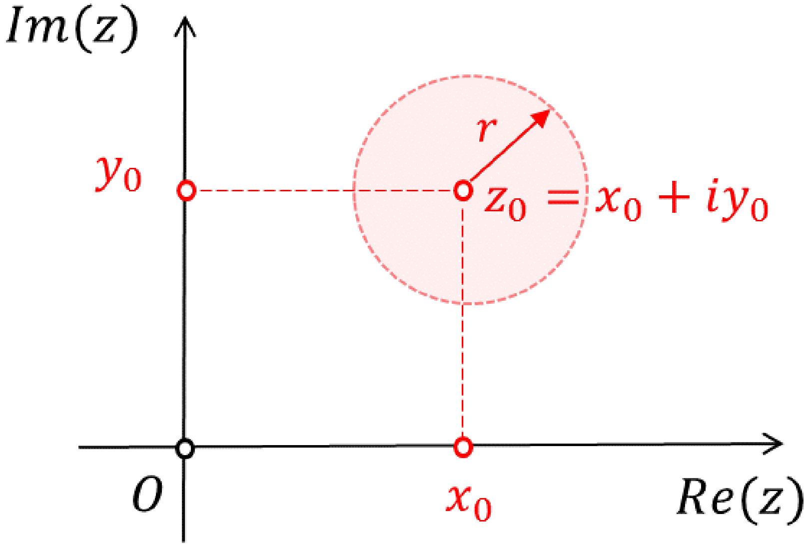
Complex-valued functions:

$$f : D \subseteq \mathbb{C} \to \mathbb{C}$$



$$z = x + iy \in D \quad \overset{f}{\longrightarrow} \quad w = u + iv$$

A special set:



Disk of radius $r > 0$ centred at z_0:

$$\{z \in \mathbb{C} : |z - z_0| < r\} \qquad (z_0 \in \mathbb{C})$$

Definition 1

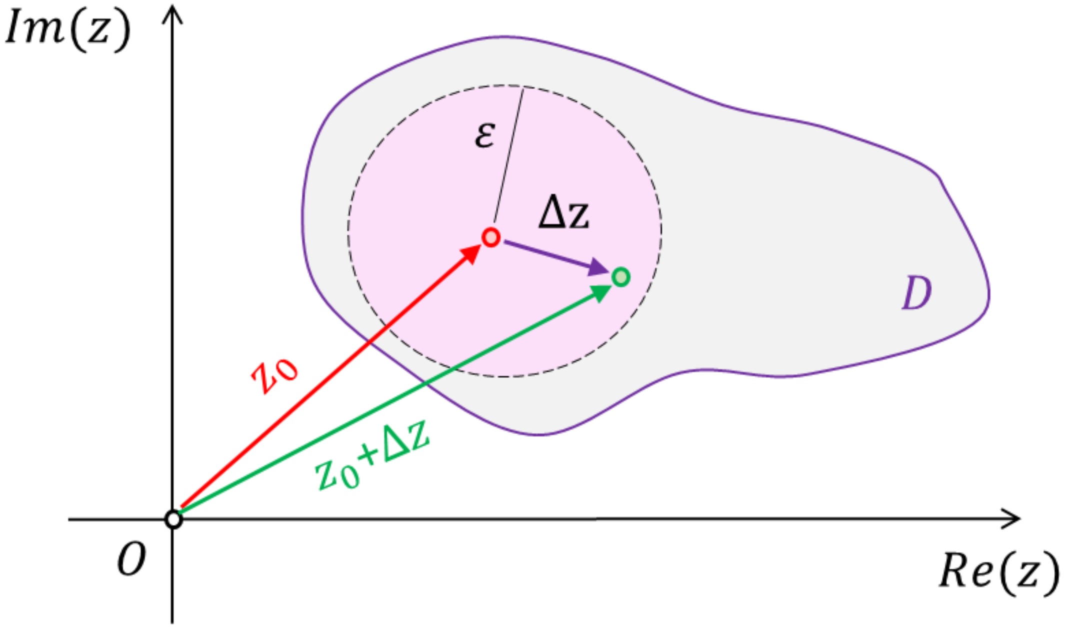
Let $w = f(z)$ be a complex-valued function whose domain contains a neighbourhood $|z - z_0| < \varepsilon$ of a point $z_0 \in \mathbb{C}$. The **derivative** of f at z_0 is defined as the limit

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{1}$$

$$\Delta z := z - z_0\,,\ (z \neq z_0)$$

$$\Rightarrow z = z_0 + \Delta z$$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{2}$$



Simplifications

- drop the subscript on z_0
- set $\underbrace{\Delta w}_{\text{change in } w = f(z)} := f(z + \Delta z) - f(z)$ and
- change $f'(z) \to \dfrac{dw}{dz}$

(2) ⇒

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

Example 1: $f(z) = z^2$

$(a+b)^2 = a^2 + 2ab + b^2$

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{z^2 + 2z\Delta z + (\Delta z)^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} (2z + \Delta z) = 2z \quad \Rightarrow \frac{dw}{dz} = 2z$$

, or $f'(z) = 2z$

We can repeat the strategy used in this example for other *power functions*:

Example 2: $f(z) = z^3$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^3 - z^3}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z^3 + 3z^2 \Delta z + 3z(\Delta z)^2 + (\Delta z)^3 - z^3}{\Delta z}$$

$$= \lim_{\Delta z \to 0} (3z^2 + 3z\Delta z + (\Delta z)^2) = 3z^2 \quad \Rightarrow \frac{dw}{dz} = 3z^2$$

$$f'(z) = 3z^2$$

In general, if $f(z) = z^n$ for some $n \in \mathbb{N}$, then $f'(z) = nz^{n-1}$

Differentiation formulas:

If the derivatives of two functions $f, g: D \to \mathbb{C}$ exist at a point $z \in \mathbb{C}$, then

$$\frac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z)$$

$$\frac{d}{dz}[\alpha f(z)] = \alpha f'(z)$$

$$\frac{d}{dz}[f(z)g(z)] = f'(z)g(z) + f(z)g'(z)$$

$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{f'(z)g(z) - f(z)g'(z)}{g^2(z)} \qquad (g(z) \neq 0)$$

These rules allow us to calculate the derivatives of simple functions, such as:

$$z^2 + z^3, \qquad z^2 + 6z^5 + 8, \qquad \frac{z^4 + 7}{z^9 + z^2 + 3}, \qquad \text{etc.}$$

Observation: We have **not** used the (very important) fact that the functions we are working with are **complex valued** and defined on subsets of the set of complex numbers $\mathbb{C}$. That is,

$$f : D \subseteq \mathbb{C} \to \mathbb{C}$$

$$z = x + iy \in D \quad \longrightarrow \quad f(z) = \underbrace{u(x,y)}_{\text{real}} + i\underbrace{v(x,y)}_{\text{imaginary}}$$

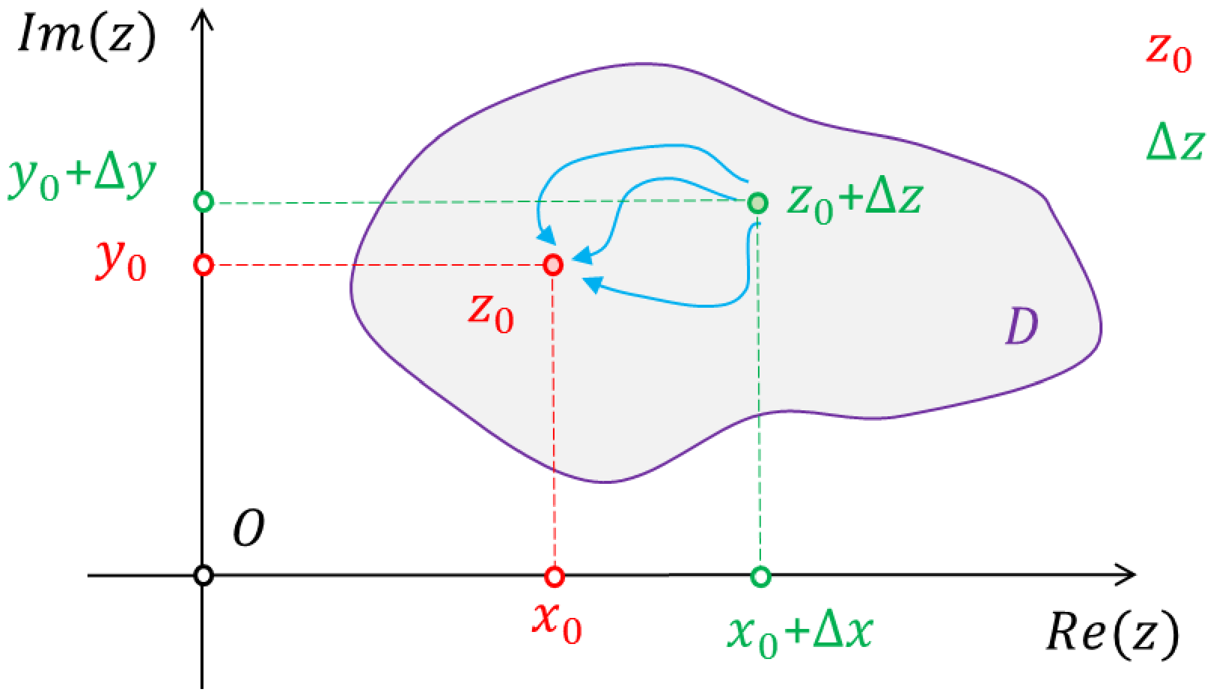
$i = \sqrt{-1}$
imaginary unit

real & imaginary parts

A closer look at the definition:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

This limit must be independent of how $\Delta z \to 0$. The point $z_0 + \Delta z$ can approach z_0 in infinitely many ways.
Three such possible paths are represented in the opposite sketch.



$z_0 = x_0 + iy_0$

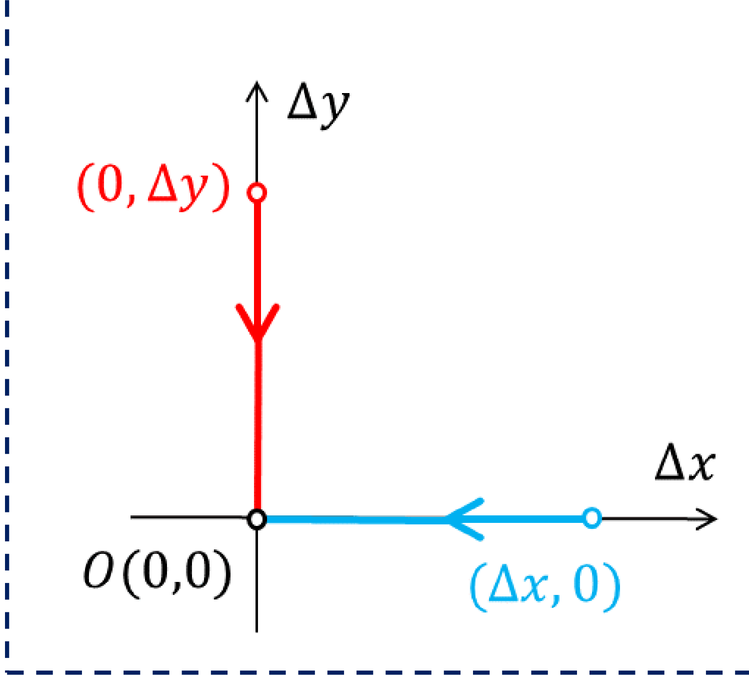
$\Delta z = \Delta x + i\Delta y$

Example 3: $f(z) = \bar{z}$

$$\frac{\Delta w}{\Delta z} = \frac{\overline{(z+\Delta z)} - \bar{z}}{\Delta z} = \frac{\bar{z} + \overline{(\Delta z)} - \bar{z}}{\Delta z} = \frac{\overline{(\Delta z)}}{\Delta z}$$

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{(\Delta z)}}{\Delta z}$$

$$\Delta z = \overbrace{(\Delta x, \Delta y)}^{\Delta x + i\Delta y} \to \overbrace{(0,0)}^{0 + i0}$$



$$\overline{\Delta z} = \overline{\Delta x + i0} = \Delta x - i0 = \Delta x + i0 = \Delta z \quad \Rightarrow \frac{\Delta w}{\Delta z} = +1$$

$$\overline{\Delta z} = \overline{0 + i\Delta y} = 0 - i\Delta y = -(0 + i\Delta y) = -\Delta z \quad \Rightarrow \frac{\Delta w}{\Delta z} = -1$$

Hence the limit of the difference quotient does NOT exist.

This means that *dw/dz does not exist anywhere.*

Example 4: $f(z) = |z|^2 = z\bar{z}$

$$\frac{\Delta w}{\Delta z} = \frac{(z+\Delta z)(\bar{z}+\overline{\Delta z}) - z\bar{z}}{\Delta z} = \bar{z} + \overline{\Delta z} + z\left(\frac{\overline{\Delta z}}{\Delta z}\right)$$

From the previous example:

$$\begin{cases} \overline{\Delta z} = \Delta z & (\Delta z \to 0\text{, horizontally}) \\ \overline{\Delta z} = -\Delta z & (\Delta z \to 0\text{, vertically}) \end{cases}$$

$$\frac{\Delta w}{\Delta z} = \begin{cases} \bar{z} + \Delta z + z & \text{when } \Delta z = (\Delta x, 0) \to (0,0) \\ \bar{z} - \Delta z - z & \text{when } \Delta z = (0, \Delta y) \to (0,0) \end{cases}$$

By the uniqueness of the limit $\lim_{\Delta z \to 0} (\Delta w / \Delta z) \Rightarrow \bar{z} + z = \bar{z} - z \quad \Rightarrow z = 0$

Conclusion: $f'(z)$ cannot exist when $z \neq 0$. At $z = 0$, $f'(0) = 0$.

General remark: (regarding the last two examples)

$$f : D \subseteq \mathbb{C} \to \mathbb{C}$$

$$z = x + iy \in D \longrightarrow \quad f(z) = u(x, y) + iv(x, y)$$

$$\left.\begin{aligned} z &= x + iy \\ \bar{z} &= x - iy \end{aligned}\right\} \text{ by adding and subtracting} \Rightarrow \left\{\begin{aligned} x &= \frac{1}{2}(z + \bar{z}) \\ y &= \frac{1}{2i}(z - \bar{z}) \end{aligned}\right.$$

Observation: In general, the expression of f depends on both z and $\bar{z}$. If the latter variable appears *explicitly* in the expression of the function, then it *cannot* be differentiable (except at some special points).

Theorem 1:

Suppose that

$$f(z) = u(x, y) + iv(x, y)$$

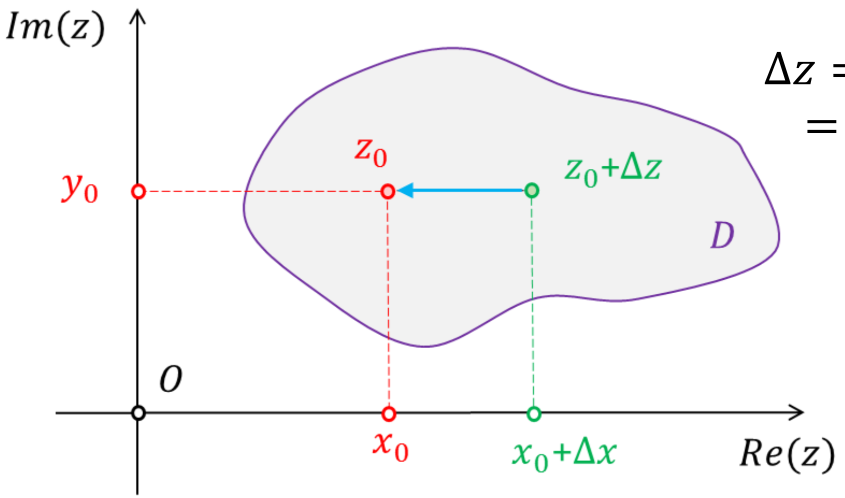
and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the first-order partial derivatives of $u(x, y)$ and $v(x, y)$ must exist at (x_0, y_0) and they must satisfy the **Cauchy-Riemann equations**:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{at } (x_0, y_0)$$

Also, the **derivative** can be calculated using the formula:

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)$$

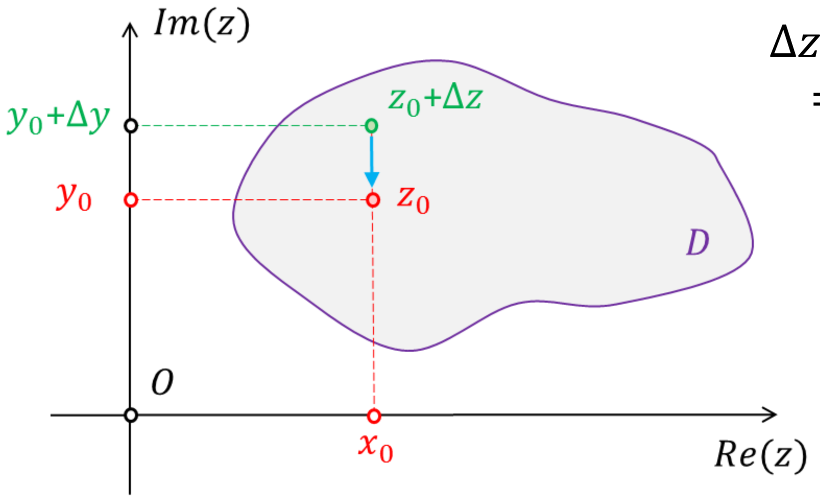
Sketch of proof: Use the difference quotient and consider the particular cases when $\Delta z \to 0$ as indicated in the two sketches below:



$$\Delta z = \Delta x + i0 = (\Delta x, 0)$$

$$\frac{\Delta w}{\Delta z} = \frac{u(x_0+\Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i\left[\frac{v(x_0+\Delta x, y_0) - v(x_0, y_0)}{\Delta x}\right]$$

$$\Rightarrow \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{\partial u}{\partial x}(x_0, y_0) + i\,\frac{\partial v}{\partial x}(x_0, y_0)$$



$$\Delta z = 0 + i\Delta y = (0, \Delta y)$$

$$\frac{\Delta w}{\Delta z} = \frac{u(x_0, y_0+\Delta y) - u(x_0, y_0)}{i\Delta y} + i\left[\frac{v(x_0, y_0+\Delta y) - v(x_0, y_0)}{i\Delta y}\right]$$

$$\Rightarrow \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{1}{i}\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0)$$

$$= -i\,\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0)$$

Equating the real and imaginary parts $\Rightarrow$ C-R equations

Example 3: $f(z) = \bar{z} = x - iy$ (revisited)

$$u(x,y) = x\,, \quad v(x,y) = -y$$

$$\frac{\partial u}{\partial x} = 1 \neq -1 = \frac{\partial v}{\partial y}$$

$\Rightarrow$ the Cauchy-Riemann equations are **not** satisfied

$\Rightarrow$ the function is **not** differentiable anywhere

Example 4: $f(z) = z\bar{z} = x^2 + y^2$ (revisited)

$$u(x,y) = x^2 + y^2\,, \qquad v(x,y) \equiv 0$$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= 2x\,, \qquad \frac{\partial v}{\partial y} = 0 \\ \frac{\partial u}{\partial y} &= 2y\,, \qquad -\frac{\partial v}{\partial x} = 0 \end{aligned}\right\}$$

the Cauchy-Riemann equations are satisfied only if $(x,y) = (0,0)$