MATHEMATICS–4H

Project code: CDC2010:2

Project title: Optimal shapes and forms.

Supervisor: Dr. C. D. Coman.

Subject area: Multivariate Calculus; Geometry.

Prerequisites: Interest in applied maths/physics; '*Mathematical modeling 1*' might help, but is not essential.

Summary: The project will deal with variational problems for 2D surfaces (Euler-Lagrange equations, for most part).

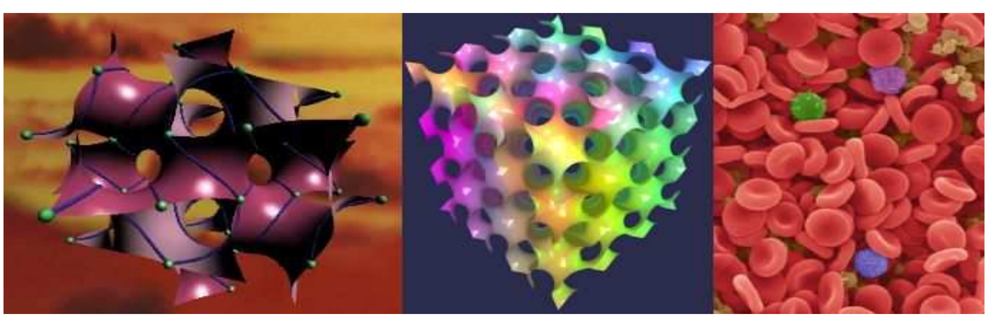
PROJECT DETAILS

The shape of a two-dimensional surface embedded in the three-dimensional Euclidean space is described, loosely speaking, by how much it bends in various directions. In mathematical terms this is achieved by considering the curvature of certain curves lying on the surface. If the surface is smooth, then at each point one can identify a pair of values, known as the principal curvatures and denoted by κ_1, κ_2; the so-called mean and Gaussian curvatures are then defined by

$$H := \frac{1}{2}(\kappa_1 + \kappa_2) \qquad \text{and} \qquad K := \kappa_1\,\kappa_2\,.$$

The project falls within the general framework of finding stationary points of surface energy functionals of the form

$$\mathcal{E}[\mathscr{S}] \equiv \iint_{\mathscr{S}} \Phi(\kappa_1, \kappa_2)\,\mathrm{d}A\,,$$



where the integrand depends symmetrically on the principal curvatures of a *variable* surface $\mathscr{S} \subset \mathbb{E}^3$. Problems of this type emerge naturally in the mathematical modeling of lipid bilayers, surfactant films, or the mechanics of thin elastic plates.

The postulated symmetry of the integrand ensures that it can in fact be represented as $\Phi \equiv \Phi(H, K)$. Relevant choices for practical problems (e.g., the shape of red blood cells – seen in the rightmost picture above) are,

$$\Phi(H, K) := \alpha + \beta(H - H_0)^2 - \gamma K \qquad (\alpha,\, \beta,\, \gamma \in \mathbb{R})$$

or

$$\Phi := (\kappa_1 + \kappa_2)^2 \equiv 4H^2\,,$$

$$\Phi := (\kappa_1 - \kappa_2)^2 \equiv 4(H^2 - K)\,,$$

$$\Phi := \kappa_1^2 + \kappa_2^2 \equiv 4H^2 - 2K\,, \quad \text{etc.}$$

For $\Phi(H, K) = H^2$, Poisson (1812) found that

$$\nabla^2 H + 2H(H^2 - K) = 0\,.$$

The purpose of the project is to obtain this equation by using multivariate calculus and some rudiments of the geometry of 3D surfaces.

General reading: *Mathematics and optimal form*, by S. Hildebrand and A. Tromba. (Gen Sci M50 1984-H).