Subject Area: Analysis

Title: Implicit functions and related problems

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Description:

Suppose that $F : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a smooth function which satisfies $F(x_0, y_0) = 0$ and $F'_y(x_0, y_0) \neq 0$, for some fixed pair of numbers $(x_0, y_0) \in \Omega$. Then, according to the *Implicit Function Theorem*, there exists a *unique* function $y \equiv y(x)$ defined locally around x_0, and such that

$$F(x, y(x)) = 0, \tag{1}$$

for all x for which the function $y(x)$ is defined. The expression of this function near x_0 can be found in the form of a power series,

$$y(x) = A_0 + A_1(x - x_0) + A_2(x - x_0)^2 + \dots, \tag{2}$$

where A_j $(j = 0, 1, 2, \dots)$ are uniquely determined by the expression of F.

If we allow for the partial derivative above to vanish at (x_0, y_0), then we lose uniqueness and the expressions of $y(x)$ satisfying (1) can no longer be found in the form indicated in (2). The problem is to determine an *explicit* expression for all possible functions $y(x)$ (as $x \to x_0$) such that condition (1) holds.

This question was first considered by Isaac Newton who looked for the solutions of (1) in the form

$$y(x) = B_0 + B_1(x - x_0)^{\alpha_1} + B_2(x - x_0)^{\alpha_2} + \dots, \tag{3}$$

where $B_j \in \mathbb{R}$ $(j = 0, 1, 2, \dots)$, and $\alpha_1 < \alpha_2 < \dots < \alpha_n < \dots$ is an increasing sequence of rational numbers (negatives allowed!). To determine the B's and the α's, Newton employed a geometric device known today as *Newton's diagram*. Further research by Lagrange and Puisseux showed that the fractional powers appearing in each series of type (3) have a finite common denominator and these series converge in a sufficiently small neighbourhood of x_0.

The main aim of the project is to provide an introduction to these ideas, and (possibly) to some applications (bifurcation theory, buckling of thin-walled mechanical structures, etc). Able students can look into the extensions of the above scenario to the case of systems of implicit equations. Also, a more complicated problem arises when one considers F as a complex-valued function of complex variables (this aspect will require knowledge of *Riemann surfaces* and the *Principle of Analytic Continuation*).

References

1. J. G. Simmonds and J. E. Mann *A First Look at Perturbation Theory*, Dover Inc., 1998.