Further Linear Elasticity

Problem Sheet #2

Torsion of cylindrical bodies

1. Consider a cylindrical body of length $L$, the ends of which are subjected to distributions of tractions that are statically equivalent to equal and opposite torques $\pm M = \pm Me_3$, the lateral surface of the cylinder being stress-free.

   (a) Explain the assumptions made by Saint-Venant (SV) in solving this problem.

   (b) Derive the approximation of the displacement field in the cylinder within the SV framework. Define the warping function $\Psi$ (also known as the Saint-Venant torsion function).

   (c) Assuming that the body force field is negligible, show that the warping function is harmonic (i.e., $\nabla^2 \Psi = 0$).

   (d) Derive the boundary conditions for the warping function.

2. Consider the setting of the previous question in the case of a circular cylinder,

   \[ \{(X_1, X_2, X_3) \in \mathbb{R}^3 \mid X_1^2 + X_2^2 \leq R^2, \ -\infty < X_3 < \infty \}. \]

   (a) Do the plane transverse cross-sections of the cylinder experience any warping? Justify your answer.

   (b) Determine the torsional rigidity of such a cylinder and show that the maximum shear stress is

   \[ \tau_{\text{max}} = \frac{2M}{\pi R^3}. \]

   [Occasionally, the torsional rigidity will be denoted by $D$.]

3. Consider again the setting of Q.1 above. Assume an elliptical cylinder defined by

   \[ \{(X_1, X_2, X_3) \in \mathbb{R}^3 \mid \frac{X_1^2}{a^2} + \frac{X_2^2}{b^2} \leq 1, \ -\infty < X_3 < \infty \} \quad (a > b). \]

   (a) Using a warping function of the form $\Psi(X_1, X_2) = kX_1X_2$, determine the value of $k \in \mathbb{R}$ for which the appropriate boundary conditions are satisfied.

   (b) Calculate the torsional rigidity of such a cylinder.

   (c) Show that the value of the maximum shear stress is

   \[ \tau_{\text{max}} = \frac{M}{\pi ab^2}. \]
(d) Find the out-of-plane displacement (responsible for the warping of the transverse cross-sections). Sketch the contours of constant warping, indicating clearly their direction relative to the longitudinal symmetry axis of the cylinder.

4. Let us consider the same situation as in Q.1, this time in the case of a cylinder whose cross-section is the rectangle defined by
\[ \Omega = \{(X_1, X_2) \in \mathbb{R}^2 \mid -a \leq X_1 \leq a, \ -b \leq X_2 \leq b\} . \]

(a) Show that the appropriate boundary conditions for the warping function are
\[ \frac{\partial \Psi}{\partial X_1} = X_2, \quad X_1 = \pm a, \]
\[ \frac{\partial \Psi}{\partial X_2} = -X_1, \quad X_2 = \pm b. \]

(b) Defining \( \Psi := X_1 X_2 - \Psi \), transform the original boundary-value problem for \( \Psi \) in terms of this new function. Use the method of separable variables to find an explicit representation for \( \Psi \).

(c) Using the standard result,
\[ \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}, \]
show that the torsional rigidity of the rectangular cylinder is given by
\[ \mu ab^3 H(\eta), \]
where
\[ \eta := \frac{a}{b} \quad \text{and} \quad H(\zeta) \equiv \frac{16\zeta^2}{3} \left[ 1 - \frac{192}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \tanh \left( \frac{(2n+1)\pi}{2\zeta} \right) \right]. \]

5. Define the Prandtl stress function \( \Phi = \Phi(X_1, X_2) \) for the torsion of non-circular cylinders. If the cross-section is a certain (possibly multiply connected) domain \( \Omega \subset \mathbb{R}^2 \) whose boundary is \( \partial \Omega \), then show that
\[ \nabla^2 \Phi = C \quad \text{in} \quad \Omega, \tag{1a} \]
\[ \frac{d\Phi}{ds} = 0 \quad \text{on} \quad \partial \Omega, \tag{1b} \]
where \( s \) represents the parameter used in describing the parametrisation of \( \partial \Omega \).

[The abbreviation PSF will be used in some of the next questions.]

6. (a) Show that the warping function \( \Psi = \Psi(X_1, X_2) \) and the Prandtl function \( \Phi = \Phi(X_1, X_2) \) are related to each other by
\[ \mu \alpha \frac{\partial \Psi}{\partial X_1} = \frac{\partial \Phi}{\partial X_2} + \mu \alpha X_2, \]
\[ \mu \alpha \frac{\partial \Psi}{\partial X_2} = -\frac{\partial \Phi}{\partial X_1} - \mu \alpha X_1, \]
where \( \alpha \) is the usual twist per unit length.
(b) Establish the formula
\[ \alpha = -\frac{1}{2\mu} \nabla^2 \Phi. \]  
(2)

[Note that according to (1a) and (2) the constants \( C \) and \( \alpha \) are related to each other; in particular \( C \neq 0 \).]

7. (a) Consider a solid cylinder as in Q.1. Show that
\[ M = 2 \int_{\Omega} \Phi \, dA, \]
where \( \Phi \) is the PSF.

\( dA \) is the usual area element; in Cartesian coordinates, \( dA = dX_1 dX_2 \), etc.

(b) Reconsider Q.3 with the help of the PSF. Start by assuming that
\[ \Phi(X_1, X_2) = m \left( \frac{X_1^2}{a^2} + \frac{X_2^2}{b^2} - 1 \right) \]
for some \( m \in \mathbb{R} \).

Next, use Q.5 – 6 to find the constant \( m \in \mathbb{R} \). Finally, show that
\[ M = \left( \frac{a^3 b^3}{a^2 + b^2} \right) \mu \alpha \pi. \]

8. The same setting as in Q.1, in which the cylinder cross-section is an equilateral triangle \( \Omega \). Assume that \( \partial \Omega = \partial \Omega^1 \cup \partial \Omega^2 \cup \partial \Omega^3 \), where
\[ \partial \Omega^1 \equiv \left\{ (X_1, X_2) \in \mathbb{R}^2 \mid X_1 - \sqrt{3} X_2 = -2a, \ -2a \leq X_2 \leq 2a \right\}, \]
\[ \partial \Omega^2 \equiv \left\{ (X_1, X_2) \in \mathbb{R}^2 \mid X_1 + \sqrt{3} X_2 = -2a, \ -2a \leq X_2 \leq 2a \right\}, \]
\[ \partial \Omega^3 \equiv \left\{ (X_1, X_2) \in \mathbb{R}^2 \mid X_1 = a, \ |X_2| \leq \sqrt{3}a \right\}. \]

(a) Explain why
\[ \Phi(X_1, X_2) = m(X_1 - \sqrt{3} X_2 + 2a)(X_1 + \sqrt{3} X_2 + 2a)(X_1 - a) \]
is an acceptable choice for the Prandtl stress function; here \( m \in \mathbb{R} \) is arbitrary.

(b) Show that the torsional rigidity of the cylinder is equal to
\[ \frac{27 \mu}{5 \sqrt{3} a^4}, \]
and find the maximal shear stress, \( \tau_{\text{max}} \).

(c) Show that the out-of-plane displacements are given by
\[ u_3(X_1, X_2) = \frac{\alpha}{6a} (3X_1^2 - X_2^2) X_2, \]
where \( \alpha \) stands for the twist per unit length.
9. Show that the torsional rigidity for a hollow cylinder of cross-section
\[ \{(r, \theta) \mid R_1 \leq r \leq R_2, \ 0 \leq \theta < 2\pi\} \quad (0 < R_1 < R_2) \]
is
\[ \frac{1}{2\mu \pi (R_2^4 - R_1^4)}. \]
If a hollow cylinder has \( R_2 = 6 \) cm and \( R_1 = 5 \) cm, find the radius of the solid cylinder which has the same torsional rigidity. Assuming that both configurations involve the same elastic material, show that the hollow cylinder is (approximately) 58% lighter than the solid cylinder.

10. (a) The torsional rigidity \( D \) of a solid cylinder of cross-section \( \Omega \subset \mathbb{R}^2 \) can be expressed in terms of the Saint-Venant torsion function as
\[
D = \mu \int_{\Omega} \left( X_1 \frac{\partial \Psi}{\partial X_2} - X_2 \frac{\partial \Psi}{\partial X_1} + X_1^2 + X_2^2 \right) \, dA. \tag{3}
\]
(b) Show that if \( u, \ v : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) are two sufficiently smooth functions (and \( \partial \Omega \) is assumed to be regular) then
\[
\int_{\Omega} (u \nabla^2 v + \nabla u \cdot \nabla v) \, dA = \oint_{\partial \Omega} u \frac{\partial v}{\partial n} \, dS, \tag{4}
\]
where
\[ \frac{\partial v}{\partial n} := n \cdot \nabla v \]
represents the normal derivative of \( v \).
(c) Using (3) and (4) establish that the torsional rigidity is always strictly positive, i.e. \( D > 0 \).

11. Same context as in Q.10 above. Show that
\[
D \leq \mu \int_{\Omega} (X_1^2 + X_2^2) \, dA;
\]
that is, the torsional rigidity of a cylinder of cross-section \( \Omega \) is never greater than \( \mu I_0 \), where \( I_0 \) is the polar moment of inertia of the domain \( \Omega \).

**Historical note:** It was Saint-Venant who first conjectured in 1853 that the torsional rigidity of a solid beam of given cross-sectional area increases as the moment of inertia of the cross-section decreases. It is also well known that circular area has the least polar moment of inertia of all simply connected regions/domains of given area. However, the mathematical proof that the circular beam indeed has the greatest torsional rigidity of all solid beams of given cross-sectional area was supplied only in 1948 by the mathematician G. Polya. In 1950, together with another pure mathematician, A. Weinstein, he generalised this neat result by proving that, of all multiply connected cross-sections with given area and with given joint area of the holes, the annulus has the maximum torsional rigidity.

12. An axisymmetric composite cylinder is composed of a solid inner shaft, of radius \( a \) and shear modulus \( \mu_1 \), and an outer sleeve of outer radius \( b \) and shear modulus \( \mu_2 \). The shaft and sleeve are ideally bonded at their interface and the composite cylinder is subjected to an applied torque \( M \).

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(a) Determine the distribution of stress within the composite cylinder in terms of the twist per unit length $\alpha$.
(b) Find an expression for the twist per unit length $\alpha$ in terms of the applied torque $M$.

13. Consider the hollow cylinder whose cross-section is bounded by two concentric, similar ellipses $\partial \Omega^1$ and $\partial \Omega^2$,

$$
\partial \Omega^1 \equiv \left\{ (X_1, X_2) \in \mathbb{R}^2 \middle| \frac{X_1^2}{a^2} + \frac{X_2^2}{b^2} = k^2 \right\},
$$
$$
\partial \Omega^2 \equiv \left\{ (X_1, X_2) \in \mathbb{R}^2 \middle| \frac{X_1^2}{a^2} + \frac{X_2^2}{b^2} = 1 \right\},
$$

where $0 < a < b$ and $0 < k < 1$ are real constants.

(a) Show that the Prandtl stress function

$$
\Phi(X_1, X_2) = m \left( \frac{X_1^2}{a^2} + \frac{X_2^2}{b^2} - 1 \right)
$$

may be used to describe the torsion for this cross-section and determine the value of the constant $m \in \mathbb{R}$ in terms of the twist per unit length $\alpha$.

(b) Find the torque $M$ as a function of twist per unit length and determine the torsional rigidity of this configuration.

14. Consider the following two disks,

$$
\mathcal{D}_1 \equiv \left\{ (X_1, X_2) \in \mathbb{R}^2 \middle| (X_1 - A)^2 + X_2^2 \leq A^2 \right\},
$$
$$
\mathcal{D}_2 \equiv \left\{ (X_1, X_2) \in \mathbb{R}^2 \middle| X_1^2 + X_2^2 \leq a^2 \right\},
$$

where $0 < a < A$. A cylindrical body whose cross-section is given by $\mathcal{D}_1 \setminus \mathcal{D}_2$ is loaded at its ends by two torsional couples $\pm M = \pm M e_3$.

(a) Considering the polar coordinates $(r, \theta) \in (0, \infty) \times [0, 2\pi)$ defined by

$$
X_1 = r \cos \theta, \quad X_2 = r \sin \theta,
$$

where

$$
r = \sqrt{X_1^2 + X_2^2}, \quad \theta = \tan^{-1} \frac{X_2}{X_1},
$$

show that

$$
\Phi(r, \theta) = C \left( r^2 - a^2 - 2Ar \cos \theta + 2 \frac{Aa^2}{r} \cos \theta \right), \quad C \in \mathbb{R}
$$

may be used as a PSF. Determine the value of the constant $C$ in terms of the twist per unit length.

(b) Calculate the stress distribution in the cross-section and find $\tau_{\text{max}}$.
(c) Find the torsional rigidity.
15. \textbf{(Leibenzon's Theorem)} The transverse cross-section of a long beam subjected to terminal torsional moments (as in Q.1) is a simply connected domain $\Omega \subset \mathbb{R}^2$ bounded by a piecewise smooth curve $\partial \Omega$. The shear stress vector is defined in the usual way,

$$\mathbf{\tau} = \sigma_3 \mathbf{e}_1 + \sigma_3 \mathbf{e}_2,$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are unit vectors along the $X_1$- and $X_2$-coordinate axes, respectively. Show that

$$\oint_{\partial \Omega} \mathbf{\tau} \cdot d\mathbf{X} = 2 \mu \alpha \text{Area}(\Omega).$$

[Note that the left-hand side of the above relation is the circulation of $\mathbf{\tau}$.]