Elasticity equations in cylindrical polar coordinates

1. The cylindrical coordinates \((r, \theta, z)\) are related to the Cartesian coordinates \((x_1, x_2, x_3)\) by the following relations

\[ r = \left( x_1^2 + x_2^2 \right)^{1/2}, \quad \theta = \tan^{-1} \frac{x_2}{x_1}, \quad z = x_3, \]
and

\[ x_1 = r \cos \theta, \quad x_2 = r \sin \theta, \quad x_3 = z. \]  \hfill (1)

The (orthogonal) base vectors in the two systems of coordinates are linked by

\[
\begin{align*}
\mathbf{e}_r &= \cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2, \\
\mathbf{e}_\theta &= -\sin \theta \mathbf{e}_1 + \cos \theta \mathbf{e}_2, \\
\mathbf{e}_z &= \mathbf{e}_3.
\end{align*}
\]

The scalar components of a vector \(\mathbf{u}\) and a second-order Cartesian tensor \(\mathbf{T}\) with respect to \(\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}\) are defined in the usual way,

\[
\begin{align*}
\mathbf{u} &= u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z, \quad \hfill (2a) \\
\mathbf{T} &= T_{rr} \mathbf{e}_r \otimes \mathbf{e}_r + T_{r\theta} \mathbf{e}_r \otimes \mathbf{e}_\theta + T_{rz} \mathbf{e}_r \otimes \mathbf{e}_z + \ldots. \quad \hfill (2b)
\end{align*}
\]

Show that

(a) if \(f\) is a scalar field, then

\[
\nabla f = \frac{\partial f}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \mathbf{e}_\theta + \frac{\partial f}{\partial z} \mathbf{e}_z; \quad \hfill (3)
\]

(b) the gradient of the vector field \(\mathbf{u}\) is given by

\[
\nabla \mathbf{u} = \frac{\partial u_r}{\partial r} \mathbf{e}_r \otimes \mathbf{e}_r + \frac{1}{r} \left( \frac{\partial u_r}{\partial \theta} - u_\theta \right) \mathbf{e}_r \otimes \mathbf{e}_\theta + \frac{\partial u_r}{\partial z} \mathbf{e}_r \otimes \mathbf{e}_z \\
+ \frac{\partial u_\theta}{\partial r} \mathbf{e}_\theta \otimes \mathbf{e}_r + \frac{1}{r} \left( \frac{\partial u_\theta}{\partial \theta} + u_r \right) \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \frac{\partial u_\theta}{\partial z} \mathbf{e}_\theta \otimes \mathbf{e}_z \\
+ \frac{\partial u_z}{\partial r} \mathbf{e}_z \otimes \mathbf{e}_r + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \mathbf{e}_z \otimes \mathbf{e}_\theta + \frac{\partial u_z}{\partial z} \mathbf{e}_z \otimes \mathbf{e}_z.
\]
2. Show that the divergence of (2b) in Question 1 is the following vector field,
\[ \nabla \cdot \mathbf{T} = (\nabla \cdot \mathbf{T})_r \mathbf{e}_r + (\nabla \cdot \mathbf{T})_\theta \mathbf{e}_\theta + (\nabla \cdot \mathbf{T})_z \mathbf{e}_z , \]
whose components are given by
\[
(\nabla \cdot \mathbf{T})_r = \frac{\partial T_{rr}}{\partial r} + \frac{1}{r} \frac{\partial T_{r\theta}}{\partial \theta} + \frac{1}{r} \frac{\partial T_{rz}}{\partial z} + \frac{1}{r} (T_{rr} - T_{\theta\theta}) ,
\]
\[
(\nabla \cdot \mathbf{T})_\theta = \frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{1}{r} \frac{\partial T_{\theta z}}{\partial z} + \frac{1}{r} (T_{r\theta} + T_{\theta r}) ,
\]
\[
(\nabla \cdot \mathbf{T})_z = \frac{\partial T_{rz}}{\partial r} + \frac{1}{r} \frac{\partial T_{\theta z}}{\partial \theta} + \frac{1}{r} T_{zr} .
\]

3. The Laplacian of a field is defined to be the divergence of its gradient, and it is usually denoted by \( \nabla^2 \) or \( \Delta \). Show that for a scalar field \( f \) expressed in cylindrical coordinates the following formula holds
\[ \nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} . \]

For a vector field \( \mathbf{u} \) the counterpart of the above formula is
\[ \nabla^2 \mathbf{u} = (\nabla^2 \mathbf{u})_r \mathbf{e}_r + (\nabla^2 \mathbf{u})_\theta \mathbf{e}_\theta + (\nabla^2 \mathbf{u})_z \mathbf{e}_z , \]
where
\[
(\nabla^2 \mathbf{u})_r = \nabla^2 u_r - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} - \frac{1}{r^2} u_r ,
\]
\[
(\nabla^2 \mathbf{u})_\theta = \nabla^2 u_\theta + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{1}{r^2} u_\theta ,
\]
\[
(\nabla^2 \mathbf{u})_z = \nabla^2 u_z .
\]

4. For a cylindrical coordinate system \((r, \theta, z)\) start from the relations (1) to obtain
\[
dx_1 = \cos \theta \, dr - r \sin \theta \, d\theta , \quad (4a)
\]
\[
dx_2 = \sin \theta \, dr + r \cos \theta \, d\theta , \quad (4b)
\]
\[
dx_3 = dz . \quad (4c)
\]
Solve for \(dr\) and \(d\theta\) in (4), to obtain
\[
\frac{dr}{dx_1} = \cos \theta , \quad \frac{dr}{dx_2} = \sin \theta , \quad \frac{d\theta}{dx_1} = -\frac{\sin \theta}{r} , \quad \frac{d\theta}{dx_2} = \frac{\cos \theta}{r} .
\]
Then show that
\[
\frac{\partial u_2}{\partial x_1} = \cos \theta \frac{\partial u_2}{\partial r} - \frac{\sin \theta}{r} \frac{\partial u_2}{\partial \theta} ,
\]
and deduce similar formulae for $\partial u_i/\partial x_j$ ($i, j = 1, 2$).

Finally, put these results together to obtain

$$\text{curl } \mathbf{u} = (\text{curl } \mathbf{u})_r \mathbf{e}_r + (\text{curl } \mathbf{u})_\theta \mathbf{e}_\theta + (\text{curl } \mathbf{u})_z \mathbf{e}_z,$$

where

$$(\text{curl } \mathbf{u})_r = \frac{1}{r} \left( \frac{\partial u_r}{\partial \theta} - r \frac{\partial u_\theta}{\partial z} \right),$$

$$(\text{curl } \mathbf{u})_\theta = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r},$$

$$(\text{curl } \mathbf{u})_z = \frac{1}{r} \left[ \frac{\partial}{\partial r} (ru_\theta) - \frac{\partial u_r}{\partial \theta} \right].$$

5. The scalar components of the displacement and (infinitesimal) strain fields in cylindrical coordinates are related to those in Cartesian coordinates by the transformation matrix

$$[Q] = \begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}$$

according to the rules

$$[\mathbf{u}]' = [Q][\mathbf{u}], \quad [\varepsilon]' = [Q][\varepsilon][Q]^T,$$

where the primed quantities are in cylindrical coordinates; in tensor notation,

$$\varepsilon = \varepsilon_{rr} \mathbf{e}_r \otimes \mathbf{e}_r + \varepsilon_{\theta\theta} \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \varepsilon_{zz} \mathbf{e}_z \otimes \mathbf{e}_z + \varepsilon_{r\theta}(\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r) + \ldots$$

e tc. Show that

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{1}{r} \left( \frac{\partial u_\theta}{\partial \theta} + u_r \right), \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z},$$

$$\varepsilon_{r\theta} = \frac{1}{2} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} \right),$$

$$\varepsilon_{\theta z} = \frac{1}{2} \left( \frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right),$$

$$\varepsilon_{zr} = \frac{1}{2} \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right).$$

6. In polar coordinates the plane stress system of equations of linear elasticity in the absence of inertial forces has the well-known form

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{1}{r} (\sigma_{rr} - \sigma_{\theta\theta}) = 0, \quad (5a)$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2}{r} \sigma_{r\theta} = 0. \quad (5b)$$
The Airy stress function $\Phi = \Phi(r, \theta)$ is introduced by demanding that

$$
\sigma_{rr} = \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2},
$$

(6a)

$$
\sigma_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2},
$$

(6b)

$$
\sigma_{r\theta} = -\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \right).
$$

(6c)

(a) Show that (5) are identically satisfied by this choice, but the compatibility relation requires that $\Phi$ is biharmonic, i.e.

$$
\nabla^4 \Phi = 0,
$$

where $\nabla^4(\bullet) = \nabla^2(\nabla^2(\bullet))$ is the bi-Laplacian and

$$
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}.
$$

(b) Establish (6) using the definition of the Airy stress function in Cartesian coordinates.

(c) Check that the following functions can be used as Airy stress functions in polar coordinates

$$
C \theta, \quad Cr^2 \theta, \quad Cr \theta \sin \theta, \quad Cr \theta \cos \theta, \quad (C \in \mathbb{R}).
$$

Furthermore, verify that $\Phi(r, \theta) = f_n(r) \cos n\theta$ and $\Phi(r, \theta) = f_n(r) \cos n\theta$ are enjoying the same property, where

$$
f_0(r) = a_0 r^2 + b_0 r^2 \log r + c_0 + d_0 \log r,
$$

$$
f_1(r) = a_1 r^3 + b_1 r + c_1 r \log r + d_1 r^{-1},
$$

$$
f_n(r) = a_n r^{n+2} + b_n r^n + c_n r^{-n+2} + d_n r^{-n}, \quad (n > 1),
$$

and $a_n, b_n, c_n, d_n \in \mathbb{R}$.

7. Consider the plane strain hollow circular shaft shown in cross-section below. A cylindrical coordinate system is defined with its origin at the centre of the cross-section. The inner surface of the shaft at $r = a$ is ideally bonded to a fixed rigid core, and the outer boundary at $r = b$ is subjected to uniform shearing traction $\tau_0$.

Using the semi-inverse method find the displacement and stress fields in the shaft.
8. A large elastic plate in equilibrium and made of an isotropic elastic material occupies the region

\[ \{(X_1, X_2) \in \mathbb{R}^2 \mid -a \leq X_1 \leq a, \quad -b \leq X_2 \leq b\} \, . \]

The plate is subjected to opposite shearing tractions \( S \) on the edges \( X_1 = \pm a \) and, respectively, \( X_2 = \pm b \), and no body forces are assumed to be present. The plate has a small hole of radius \( r = a \) and centre \((0,0)\).

If the plate is very large (strictly speaking \( a = \infty \) and \( b = \infty \)), show that the local stress field near the hole admits the polar coordinate representation

\[
\sigma_{rr}(r, \theta) = S \left[ 1 - 4 \left( \frac{a}{r} \right)^2 + 3 \left( \frac{a}{r} \right)^4 \right] \sin 2\theta ,
\]

\[
\sigma_{\theta\theta}(r, \theta) = -S \left[ 1 + 3 \left( \frac{a}{r} \right)^4 \right] \sin 2\theta ,
\]

\[
\sigma_{r\theta}(r, \theta) = S \left[ 1 + 2 \left( \frac{a}{r} \right)^2 - 3 \left( \frac{a}{r} \right)^4 \right] \cos 2\theta .
\]

9. (The problem of Kirsch, 1898) Consider again an elastic plate in equilibrium. The plate occupies the region

\[ \{(X_1, X_2) \in \mathbb{R}^2 \mid -\infty \leq X_1 \leq \infty , \quad -b \leq X_2 \leq b\} \, . \]

The plate is submitted to a uniform tension of magnitude \( T \) in the \( X_1 \)–direction, i.e. \( \sigma_{11}(X_1, X_2) \to T \) as \( X_1 \to \pm \infty \). A small hole of radius \( r = a \) is made in the middle of the plate, where \( a \ll b \). Using an Airy stress function of the form

\[ \Phi(r, \theta) = \left( Ar^2 + Br^4 + \frac{C}{r^2} + D \right) \cos 2\theta , \]

show that the local stress distribution near the central hole is given by

\[
\sigma_{rr}(r, \theta) = \frac{T}{2} \left[ 1 - \left( \frac{a}{r} \right)^2 \right] + \frac{T}{2} \left[ 1 - 4 \left( \frac{a}{r} \right)^2 + 3 \left( \frac{a}{r} \right)^4 \right] \cos 2\theta ,
\]

\[
\sigma_{\theta\theta}(r, \theta) = \frac{T}{2} \left[ 1 + \left( \frac{a}{r} \right)^2 \right] - \frac{T}{2} \left[ 1 + 3 \left( \frac{a}{r} \right)^4 \right] \cos 2\theta ,
\]

\[
\sigma_{r\theta}(r, \theta) = -\frac{T}{2} \left[ 1 + 2 \left( \frac{a}{r} \right)^2 - 3 \left( \frac{a}{r} \right)^4 \right] \sin 2\theta .
\]
10. A thin elastic plate in the shape of a circular annulus, 

\[ \{(r, \theta) \mid R_1 \leq r \leq R_2, \ 0 \leq \theta < 2\pi\} \]

is subjected to in-plane radial displacements as indicated in the figure below.

(a) Show that under the \textit{axial symmetry assumption}, the stress distribution in the stretched configuration assumes the form

\begin{align*}
\sigma_{rr} &= \frac{E}{1 - \nu^2} \left[ (1 + \nu) \frac{U_1 R_1 + U_2 R_2}{R_2^2 - R_1^2} + (1 - \nu) \frac{U_1 R_2 + U_2 R_1}{R_2^2 - R_1^2} \left( \frac{R_1 R_2}{r^2} \right) \right], \quad (7a) \\
\sigma_{\theta\theta} &= \frac{E}{1 - \nu^2} \left[ (1 + \nu) \frac{U_1 R_1 + U_2 R_2}{R_2^2 - R_1^2} - (1 - \nu) \frac{U_1 R_2 + U_2 R_1}{R_2^2 - R_1^2} \left( \frac{R_1 R_2}{r^2} \right) \right], \quad (7b)
\end{align*}

where \( \nu \) is the Poisson’s ratio and \( E \) stands for the Young’s modulus characterising the elastic material of the plate.

(b) Let

\[ \lambda := \frac{U_1}{U_2}, \quad \eta := \frac{R_1}{R_2}, \quad \overline{\eta} := \sqrt{\frac{1 - \nu}{1 + \nu}}. \]

Prove that if \( \eta < \overline{\eta} \), then the azimuthal stresses (7b) become compressive in the annular region

\[ \{(r, \theta) \mid R_1 \leq r \leq R_2 \overline{\rho}, \ 0 \leq \theta < 2\pi\}, \]

where

\[ \overline{\rho} := \sqrt{\left( \frac{1 - \nu}{1 + \nu} \right) \eta^2 + \frac{\lambda \eta}{1 + \lambda \eta}}. \quad (8) \]

Find the minimum value of \( \lambda \) that corresponds to the onset of compressive stresses in the annulus. Are there any compressive stresses in the plate when \( \eta > \overline{\eta} \)?

11. An annular elastic plate of inner radius \( R_1 \), outer radius \( R_2 \) and thickness \( h \) \((h/R_2 \ll 1)\) is initially stretched by imposing the uniform displacement field \( U_0 > 0 \) along the outer edge, \( r = R_2 \), while the inner boundary, \( r = R_1 \), is rotated through a small angle by applying a torque \( M \) (see the figure included below).
This in-plane rotation is achieved by means of a rigid shaft.

Using a cylindrical system of coordinates \((r, \theta, z)\) defined in an obvious manner, show that the axisymmetric stress distribution in the plate is given by

\[
\sigma_{rr}(r) = \frac{E}{1 + \nu} \left( \frac{U_0 R_2}{R_2^2 - R_1^2} \right) \left[ \frac{1 + \nu}{1 - \nu} \frac{R_2^2}{r^2} \right], \tag{9}
\]

\[
\sigma_{\theta\theta}(r) = \frac{E}{1 + \nu} \left( \frac{U_0 R_2}{R_2^2 - R_1^2} \right) \left[ \frac{1 + \nu}{1 - \nu} \frac{R_2^2}{r^2} \right], \tag{10}
\]

\[
\sigma_{r\theta}(r) = -\left( \frac{M}{2\pi h} \right) \frac{1}{r^2}. \tag{11}
\]

Investigate further whether there are any directions in which compressive stresses appear in the plate.

12. The curved beam shown below occupies the region

\[
\left\{ (r, \theta) \mid a \leq r \leq b, \ 0 \leq \theta < \frac{\pi}{2} \right\}
\]

and is loaded by a statically equivalent bending moment \(M_0\).
Assuming that the curved boundaries are stress free (i.e., \( \sigma_{rr} = \sigma_{r\theta} = 0 \) for \( r = R_{1,2} \)) and the beam is in a state of plane stress/strain, use the Airy stress function

\[
\Phi(r, \theta) = Ar^2 + Br^2 \log r + C \log r + D\theta, \quad (A, B, C, D \in \mathbb{R})
\]

\[\text{to find an approximate distribution of stresses in the beam (within the bounds of the Saint-Venant theory).}\]

13. The curved beam seen below is curved along a circular arc. The beam is fixed at the upper end and subject at the lower end to a distribution of tractions statically equivalent to a force per thickness \( F = -Fe_1 \).

![Diagram of curved beam](image)

Assume that the beam is in a state of plane stress/strain. Show that an Airy stress function of the form

\[
\Phi(r, \theta) = \left( Ar^3 + \frac{B}{r} + Cr \log r \right) \sin \theta
\]

provides an approximate solution to this problem (in the sense of Saint-Venant’s Principle), and solve for the values of the real constants \( A, B, C \).