Intro to Continuum Mechanics 2018
(solutions for the coursework)

Q.1 a). Let $\underset{\sim}{T} := (\underline{a} \otimes \underline{b} + \underline{c} \otimes \underline{d})^*$

$\underset{\sim}{S} := (\underline{a} \wedge \underline{c}) \otimes (\underline{b} \wedge \underline{d})$

Must show that $\underset{\sim}{T}(\underline{u} \wedge \underline{v}) = \underset{\sim}{S}(\underline{u} \wedge \underline{v}) \quad (\forall)\ \underline{u}, \underline{v} \in \mathcal{V}$

$$\text{LHS} = (\underline{a} \otimes \underline{b} + \underline{c} \otimes \underline{d})^* (\underline{u} \wedge \underline{v}) \underset{\text{def of } *}{=} [(\underline{a} \otimes \underline{b} + \underline{c} \otimes \underline{d})\underline{u}] \wedge [(\underline{a} \otimes \underline{b} + \underline{c} \otimes \underline{d})\underline{v}]$$

$$= [\underline{a}(\underline{b} \cdot \underline{u}) + \underline{c}(\underline{d} \cdot \underline{u})] \wedge [\underline{a}(\underline{b} \cdot \underline{v}) + \underline{c}(\underline{d} \cdot \underline{v})]$$

$$= (\underline{a} \wedge \underline{c})(\underline{b} \cdot \underline{u})(\underline{d} \cdot \underline{v}) + (\underline{c} \wedge \underline{a})(\underline{d} \cdot \underline{u})(\underline{b} \cdot \underline{v})$$

$$= (\underline{a} \wedge \underline{c})[(\underline{b} \cdot \underline{u})(\underline{d} \cdot \underline{v}) - (\underline{d} \cdot \underline{u})(\underline{b} \cdot \underline{v})]$$

$$\overset{(I)}{=} (\underline{a} \wedge \underline{c})[(\underline{b} \wedge \underline{d}) \cdot (\underline{u} \wedge \underline{v})] \quad (1)$$

$$\text{RHS} = [(\underline{a} \wedge \underline{c}) \otimes (\underline{b} \wedge \underline{d})](\underline{u} \wedge \underline{v}) = (\underline{a} \wedge \underline{c})[(\underline{b} \wedge \underline{d}) \cdot (\underline{u} \wedge \underline{v})] \quad (2)$$

OBS. In (I) I used the formula

$$(\underline{A} \wedge \underline{B}) \cdot (\underline{C} \wedge \underline{D}) = \begin{vmatrix} \underline{A} \cdot \underline{C} & \underline{A} \cdot \underline{D} \\ \underline{B} \cdot \underline{C} & \underline{B} \cdot \underline{D} \end{vmatrix} \qquad \underline{A}, \underline{B}, \underline{C}, \underline{D} = \text{vectors}$$

Comparing (1) and (2) $\Rightarrow$ QED

b). If $\underline{a}, \underline{b}, \underline{c}$ are orthonormal $\Rightarrow \underline{a} \wedge \underline{b} = \underline{c}$, $\underline{b} \wedge \underline{c} = \underline{a}$, $\underline{c} \wedge \underline{a} = \underline{b}$

and these vectors have unit magnitude

Take $\underline{d} = \underline{c}$ in the identity proved above

$$(\underline{a}\otimes\underline{b}+\underline{c}\otimes\underline{c})^* = \underbrace{(\underline{a}\wedge\underline{c})}_{-\underline{b}}\otimes\underbrace{(\underline{b}\wedge\underline{c})}_{\underline{a}} \Rightarrow \underline{b}\otimes\underline{a} = -(\underline{a}\otimes\underline{b}+\underline{c}\otimes\underline{c})^*$$

QED

b). Let $\underline{v}, \underline{w}$ be two unit vectors such that $\{\underline{u}, \underline{v}, \underline{w}\}$ forms an orthonormal system

Then $\underset{\sim}{I} = \underline{u}\otimes\underline{u}+\underline{v}\otimes\underline{v}+\underline{w}\otimes\underline{w}$

$$\Rightarrow \underset{\sim}{I} - \underline{u}\otimes\underline{u} = \underline{v}\otimes\underline{v}+\underline{w}\otimes\underline{w}$$

(1) in Q1

$$\Rightarrow (\underset{\sim}{I}-\underline{u}\otimes\underline{u})^* = (\underline{v}\otimes\underline{v}+\underline{w}\otimes\underline{w})^* = (\underline{v}\wedge\underline{w})\otimes(\underline{v}\wedge\underline{w})$$
$$= \underline{u}\otimes\underline{u}$$

If $|\underline{u}| \neq 1$ then let $\underline{\hat{u}} := \frac{\underline{u}}{|\underline{u}|}$ which is a unit vector

Consider again $\underline{v}, \underline{w}$ s.t. $\{\underline{\hat{u}}, \underline{v}, \underline{w}\}$ is an orthonormal system

$$\underset{\sim}{I}-\underline{u}\otimes\underline{u} = \underline{\hat{u}}\otimes\underline{\hat{u}}+\underline{v}\otimes\underline{v}+\underline{w}\otimes\underline{w} - |\underline{u}|^2\,\underline{\hat{u}}\otimes\underline{\hat{u}}$$
$$= (1-|\underline{u}|^2)\,\underline{\hat{u}}\otimes\underline{\hat{u}}+\underline{v}\otimes\underline{v}+\underline{w}\otimes\underline{w} \qquad (*)$$

[OBS. if $\underset{\sim}{A}\in CT(2) \Rightarrow \underset{\sim}{A}^* = (\det\underset{\sim}{A})\,\underset{\sim}{A}^{-T}$]

Let $\underset{\sim}{A}$ be the RHS of $(*)$.

$$\det\underset{\sim}{A} = (1-|\underline{u}|^2)\cdot 1\cdot 1 = 1-|\underline{u}|^2$$

$$\underset{\sim}{A}^T = \underset{\sim}{A} \quad \text{and} \quad \underset{\sim}{A}^{-1} = \frac{1}{1-|\underline{u}|^2}\,\underline{\hat{u}}\otimes\underline{\hat{u}}+\underline{v}\otimes\underline{v}+\underline{w}\otimes\underline{w}$$

$$(\underset{\sim}{I}-\underline{u}\otimes\underline{u})^* = (1-|\underline{u}|^2)\left[\frac{1}{1-|\underline{u}|^2}\,\underline{\hat{u}}\otimes\underline{\hat{u}}+\underline{v}\otimes\underline{v}+\underline{w}\otimes\underline{w}\right]$$

$$= \hat{\underline{u}} \otimes \hat{\underline{u}} + (1-|\underline{u}|^2)(\underline{v}\otimes\underline{v} + \underline{w}\otimes\underline{w} + \hat{\underline{u}}\otimes\hat{\underline{u}} - \hat{\underline{u}}\otimes\hat{\underline{u}})$$

$$= \hat{\underline{u}} \otimes \hat{\underline{u}} + (1-|\underline{u}|^2)\,\underset{\sim}{I} - \hat{\underline{u}}\otimes\hat{\underline{u}} + \hat{\underline{u}}\otimes\hat{\underline{u}}\,|\underline{u}|^2$$

$$\Rightarrow \boxed{\left(\underset{\sim}{I} - \underline{u}\otimes\underline{u}\right)^{*} = (1-|\underline{u}|^2)\,\underset{\sim}{I} + \underline{u}\otimes\underline{u}} \qquad \text{if } |\underline{u}| \neq 1$$

c). We use one of the solved examples:

$$\underset{\sim}{A} = \underset{\sim}{A}(\tau) \qquad \text{2nd order tensor} \qquad \tau = \text{parameter}$$

$$\frac{d}{d\tau}\left(\det \underset{\sim}{A}\right) = \left(\det \underset{\sim}{A}\right) \operatorname{tr}\left[\frac{d\underset{\sim}{A}}{d\tau}\,\underset{\sim}{A}^{-1}\right]$$

Take $\underset{\sim}{A} \to \underset{\sim}{F}$ (def grad) $\qquad \tau \to X_\alpha$

$$\frac{\partial J}{\partial X_\alpha} = J \operatorname{tr}\left(\frac{\partial \underset{\sim}{F}}{\partial X_\alpha}\,\underset{\sim}{F}^{-1}\right) = J\,\frac{\partial F_{p\gamma}}{\partial X_\alpha}\,F^{-1}_{\gamma p} = J\,\frac{\partial^2 x_p}{\partial X_\alpha \partial X_\gamma}\,F^{-1}_{\gamma p}$$

$$\frac{\partial}{\partial x_p}\left(J^{-1}F_{p\alpha}\right) = \frac{\partial}{\partial X_\beta}\left(J^{-1}F_{p\alpha}\right)\frac{\partial X_\beta}{\partial x_p} \to F^{-1}_{\beta p}$$

$$= \left(\frac{\partial}{\partial X_\beta}(J^{-1})\,F_{p\alpha} + J^{-1}\frac{\partial F_{p\alpha}}{\partial X_\beta}\right)F^{-1}_{\beta p}$$

$$= \left(J^{-1}\frac{\partial^2 x_p}{\partial X_\alpha \partial X_\beta} - J^{-1}\frac{\partial^2 x_q}{\partial X_\beta \partial X_\delta}\,F^{-1}_{\delta q}\,\underbrace{F_{p\alpha}\right)F^{-1}_{\beta p}}_{\delta_{\alpha\beta}}$$

$$= J^{-1}\left(\frac{\partial^2 x_q}{\partial X_\alpha \partial X_\delta}\,F^{-1}_{\delta q} - \frac{\partial^2 x_q}{\partial X_\beta \partial X_\delta}\,\delta_{\beta\alpha}\,F^{-1}_{\delta q}\right)$$

$$= 0$$

d). $\overset{\circ}{\sigma} = \dot{\sigma} - L\sigma - \sigma L^T + \sigma\, tr(L)$

Solved examples $\rightarrow$ $\dot{\sigma} + L^T\sigma + \sigma L$ is objective $\rightarrow$ result remains true for $\dot{\sigma} - L\sigma - \sigma L^T$

It remains to show that $\sigma\, tr(L)$ is objective. Let $K := \sigma\, tr(L)$

Must show that

$$K^* = Q K Q^T$$

$$K^* = \sigma^* tr(L^*) = \sigma^* tr[\underbrace{\dot{Q}Q^T}_{\text{skew symmetric} \rightarrow \text{trace}=0} + QLQ^T] \overset{\sigma \text{ objective}}{=} (Q\sigma Q^T)\, \underbrace{tr(QLQ^T)}_{tr(L)}$$

$$= Q[\sigma\, tr(L)]Q^T = QKQ^T \quad \text{QED.}$$

Q.2 a) $(a\otimes b)(a\otimes x) = (x\otimes b)(a\otimes a)$ $\qquad a\cdot b \neq 0$

Using the properties in Chapter 1

$$LHS = (a\cdot b)(a\otimes x) \qquad RHS = (a\cdot b)(x\otimes a)$$

So the equation becomes $a\otimes x = x\otimes a \quad \Big|\cdot a$

$$\Rightarrow (a\otimes x)a = (x\otimes a)a \iff (x\cdot a)a = (a\cdot a)x$$

$$\text{so } x = \lambda a \qquad (\lambda\in\mathbb{R})$$

b). Use the Divergence Theorem (Chapter 1):

$$\int_{\partial\Omega} u\otimes(T^T n)\, da = \int_{\Omega}\left[u\otimes(\operatorname{div} T) + (\operatorname{grad} u)\, T\right] dv$$

$\Omega \to \mathcal{B}_c \qquad \mathbf{u} \to \mathbf{x} \qquad \mathbf{I} \to \boldsymbol{\sigma}$

$$\int_{\partial \mathcal{B}_c} \mathbf{x} \otimes (\boldsymbol{\sigma}^T \mathbf{n})\, da = \int_{\mathcal{B}_c} \mathbf{x} \otimes \operatorname{div} \boldsymbol{\sigma}\, dv + \int_{\mathcal{B}_c} \boldsymbol{\sigma}\, dv \qquad \left(\text{since } \operatorname{grad} \mathbf{x} = \mathbf{I}\right)$$

equilibrium eqns. $\operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{b} = 0$

$\Rightarrow \operatorname{div} \boldsymbol{\sigma} = -\rho \mathbf{b}$

$\boldsymbol{\sigma}$ symmetric

$$\int_{\partial \mathcal{B}_c} \mathbf{x} \otimes (\boldsymbol{\sigma} \mathbf{n})\, da = -\int_{\mathcal{B}_c} \rho\, \mathbf{x} \otimes \mathbf{b}\, dv + \int_{\mathcal{B}_c} \boldsymbol{\sigma}\, dv$$

Transpose this relation and use $(\mathbf{a} \otimes \mathbf{b})^T = \mathbf{b} \otimes \mathbf{a}$, $\mathbf{a}, \mathbf{b}$ = vectors

$$\Rightarrow \int_{\mathcal{B}_c} \boldsymbol{\sigma}\, dv = \int_{\partial \mathcal{B}_c} (\boldsymbol{\sigma} \mathbf{n}) \otimes \mathbf{x}\, da + \int_{\mathcal{B}_c} \rho\, (\mathbf{b} \otimes \mathbf{x})\, dv \quad \text{QED}$$

c). Nanson's Formula: $\mathbf{n}\, da = J \mathbf{F}^{-T} \mathbf{N}\, dA$

Take the dot product with itself:

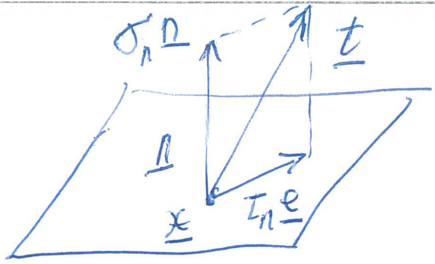
$$(da)^2 = (J \mathbf{F}^{-T} \mathbf{N}) \cdot (J \mathbf{F}^{-T} \mathbf{N}) (dA)^2$$

$$= J^2 \left[(\mathbf{F}^{-T} \mathbf{N}) \cdot (\mathbf{F}^{-T} \mathbf{N}) \right] (dA)^2$$

$$= J^2 \left[\mathbf{N} \cdot (\mathbf{F}^{-1} \mathbf{F}^{-T} \mathbf{N}) \right] (dA)^2 = J^2 (\mathbf{N} \cdot \mathbf{C}^{-1} \mathbf{N}) (dA)^2$$

$$\mathbf{C} = \mathbf{F}^T \mathbf{F} \Rightarrow \mathbf{C}^{-1} = (\mathbf{F}^T \mathbf{F})^{-1} = \mathbf{F}^{-1} \mathbf{F}^{-T}$$

$$\Rightarrow \left(\frac{da}{dA}\right)^2 = J^2\, \mathbf{N} \cdot (\mathbf{C}^{-1} \mathbf{N}) \Rightarrow \frac{da}{dA} = J \sqrt{\mathbf{N} \cdot (\mathbf{C}^{-1} \mathbf{N})} \qquad \text{QED}$$



d). $\underset{\sim}{\sigma}\,\underline{n} - (\underline{n}\cdot\underset{\sim}{\sigma}\,\underline{n})\,\underline{n}$ = shearing force

Let $\underline{t} = \sigma_n \underline{n} + \tau_n \underline{e}$

$|\underline{n}| = |\underline{e}| = 1$

$$\tau_n^2 = |\underline{t}|^2 - \sigma_n^2 \qquad (1)$$

Let $\underline{m}_1, \underline{m}_2, \underline{m}_3$ be the principal directions of $\underset{\sim}{\sigma}$ $\qquad \sigma_1, \sigma_2, \sigma_3$ = principal stresses

$\underset{\sim}{\sigma}$ symmetric $\to$ can choose $\underline{m}_1, \underline{m}_2, \underline{m}_3$ s.t. they form an orthonormal system

$$\underset{\sim}{\sigma} = \sigma_1\, \underline{m}_1 \otimes \underline{m}_1 + \sigma_2\, \underline{m}_2 \otimes \underline{m}_2 + \sigma_3\, \underline{m}_3 \otimes \underline{m}_3 \qquad (2)$$

$$\underline{n} = n_1 \underline{m}_1 + n_2 \underline{m}_2 + n_3 \underline{m}_3 \qquad (3)$$

Then $\underset{\sim}{\sigma}\,\underline{n} = \underline{t} \Rightarrow \quad \underline{t} = \sigma_1 n_1 \underline{m}_1 + \sigma_2 n_2 \underline{m}_2 + \sigma_3 n_3 \underline{m}_3 \quad (4)$

From (1), (2), (3), (4) it follows that

$$\tau_n^2 = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 - (\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2)^2$$

Want to find the extremal values of τ_n^2 subject to

$$(5) \qquad n_1^2 + n_2^2 + n_3^2 = 1 \qquad (\Leftrightarrow |\underline{n}| = 1)$$

Use the Method of the Lagrange Multipliers :

Define $\quad \Phi_n := \tau_n^2 - \lambda(n_1^2 + n_2^2 + n_3^2 - 1)$

Stationary points of Φ_n : $\quad \dfrac{\partial \Phi_n}{\partial n_1} = \dfrac{\partial \Phi_n}{\partial n_2} = \dfrac{\partial \Phi_n}{\partial n_3} = 0 = \dfrac{\partial \Phi_n}{\partial \lambda}$

Note that $\dfrac{\partial \Phi_n}{\partial \lambda} = 0 \Rightarrow (5)$.

The other 3 equations produce $\quad n_j \left[\sigma_j^2 - 2(\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2)\sigma_j - \lambda\right] = 0$

(no summation) $\qquad j = 1, 2, 3 \qquad (6)$

For simplicity assume that $\sigma_3 < \sigma_2 < \sigma_1$ (7)

Clearly at least one of the n_j's $(j=1,2,3)$ must be zero

Assume by contradiction that $n_j \neq 0$ $(\forall)\ j=1,2,3$

$$\Rightarrow \quad \sigma_j^2 - 2\left(\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2\right)\sigma_j - \lambda = 0 \qquad j=1,2,3$$

Take the difference of the first two and the last two equations

$$\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 = \frac{1}{2}\left(\sigma_1+\sigma_2\right) = \frac{1}{2}\left(\sigma_2+\sigma_3\right) \qquad (8)$$

(7) & (8) $\Rightarrow$ contradiction

Assume that $n_1 = 0$ $\quad n_2 \neq 0, \quad n_3 \neq 0$

From (6) $\Rightarrow$ $\left.\begin{aligned} &\sigma_2 n_2^2 + \sigma_3 n_3^2 = \frac{1}{2}(\sigma_2+\sigma_3) \\ &\text{also } \ n_2^2 + n_3^2 = 1 \end{aligned}\right\} \Rightarrow |n_1| = |n_2| = \frac{\sqrt{2}}{2}$

In this case $\tau_n = \pm \frac{1}{2}\left(\sigma_2 - \sigma_3\right)$

In a similar way one can discuss the cases

$$n_1 \neq 0 \quad n_2 = 0 \quad n_3 \neq 0$$
$$n_1 \neq 0 \quad n_2 \neq 0 \quad n_3 = 0$$

There are 3 extremal values of τ_n denoted by τ_1, τ_2, τ_3

$$\tau_1 = \frac{1}{2}\left(\sigma_2 - \sigma_3\right) \qquad \tau_2 = \frac{1}{2}\left(\sigma_1 - \sigma_3\right) \qquad \tau_3 = \frac{1}{2}\left(\sigma_1 - \sigma_2\right)$$

acting on planes with unit normals given by

$$\frac{1}{\sqrt{2}}\left(\pm \underline{m}_3 \pm \underline{m}_2\right) \qquad \frac{1}{\sqrt{2}}\left(\pm \underline{m}_3 \pm \underline{m}_1\right) \qquad \frac{1}{\sqrt{2}}\left(\pm \underline{m}_1 \pm \underline{m}_2\right)$$

By considering the cases when 2 components of $\underline{n}$ are zero

$$n_1 = 0 \qquad n_2 = 0 \qquad n_3 \neq 0$$
$$n_1 \neq 0 \qquad n_2 = 0 \qquad n_3 = 0$$
$$n_1 = 0 \qquad n_2 \neq 0 \qquad n_3 = 0$$

We get $\tau_n = 0$. Hence the shear stress is minimum (and equal to zero) when $\underline{n}$ coincides with the principal directions of $\underset{\sim}{\sigma}$ (look at (3))

Q.3 a).

$$\begin{vmatrix} 3-\lambda & 2 & 2 \\ 2 & 4-\lambda & 0 \\ 2 & 0 & 2-\lambda \end{vmatrix} = -4(4-\lambda) + (2-\lambda)\left[(3-\lambda)(4-\lambda)-4\right]$$

expand along this row

$$= 4(\lambda-4) + (2-\lambda)(\lambda^2-7\lambda+8)$$
$$= 4\lambda - 16 + 2\lambda^2 - 14\lambda + 16 - \lambda^3 + 7\lambda^2 - 8\lambda$$
$$= -\lambda^3 + 9\lambda^2 - 18\lambda$$

Eigenvalues: $\lambda(\lambda^2 - 9\lambda + 18) = 0 \rightarrow 0, 3, 6$

orthonormal eigenvectors:

$6 \qquad \frac{1}{3}(-2,-2,-1) = \underline{m}_1$

$3 \qquad \frac{1}{3}(1,-2,2) = \underline{m}_2$

$0 \qquad \frac{1}{3}(-2,1,2) = \underline{m}_3$

principal axes

By direct calculations $\underline{m}_1 \cdot \underline{m}_2 = \underline{m}_2 \cdot \underline{m}_3 = \underline{m}_3 \cdot \underline{m}_1 = 0$ and $|\underline{m}_1| = |\underline{m}_2| = |\underline{m}_3| = 1$

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & -2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow [\underset{\sim}{Q}] = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & -2 & -1 \end{bmatrix}$$

$$[\underset{\sim}{Q}^T] = \frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ 2 & 2 & -1 \end{bmatrix}$$

$$[\underset{\sim}{Q}][\underset{\sim}{Q}^T] = \frac{1}{9}\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & -2 & -1 \end{bmatrix}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ 2 & 2 & -1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1+4+4 & -2-2+4 & -2+4-2 \\ -2-2+4 & 4+1+4 & 4-2-2 \\ -2+4-2 & 4-2-2 & 4+4+1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i.e. $\underset{\sim}{Q}$ is orthogonal

$$[\hat{\underset{\sim}{\sigma}}] = [\underset{\sim}{Q}][\underset{\sim}{\sigma}][\underset{\sim}{Q}^T] \Rightarrow [\hat{\underset{\sim}{\sigma}}] = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix} \rightarrow \underset{\sim}{\sigma} \text{ has a diagonal form}$$

↑ new $\underset{\sim}{\sigma}$

$$\underline{t} = \underset{\sim}{\sigma}\,\underline{n}$$

plane normal to the x_1-axis $\quad \underline{n} = (\pm 1, 0, 0)$

$$\underline{t} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \pm 3 \\ \pm 2 \\ \pm 2 \end{bmatrix} \Rightarrow \underline{t} = \pm\left(3\underline{e}_1 + 2\underline{e}_2 + 2\underline{e}_3\right)$$

plane whose normal has direction ratios: $\quad 1 : -3 : 2 \Rightarrow \underline{n} = \frac{\pm 1}{\sqrt{14}}\left(\underline{e}_1 - 3\underline{e}_2 + 2\underline{e}_3\right)$

$$\Rightarrow \underline{t} = \pm \frac{1}{\sqrt{14}}\left(\underline{e}_1 - 10\underline{e}_2 + 6\underline{e}_3\right)$$

plane parallel to $x_1 + 2x_2 + 3x_3 = 1$ $\Rightarrow \underline{n} = \pm \frac{1}{\sqrt{14}}\left(\underline{e}_1 + 2\underline{e}_2 + 3\underline{e}_3\right)$

$$\Rightarrow \underline{t} = \pm \frac{1}{\sqrt{14}}\left(13\underline{e}_1 + 10\underline{e}_2 + 8\underline{e}_3\right)$$

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b). $\underset{\sim}{C} = \underset{\sim}{F}^T \underset{\sim}{F}$ $\rightarrow$ calculate $\underset{\sim}{F}$ $\rightarrow$ take the product of $\underset{\sim}{F}^T$ and $\underset{\sim}{F}$, etc

$$\underset{\sim}{C} = \begin{bmatrix} 2\alpha^2 & 0 & 0 \\ 0 & 2\beta^2 & 0 \\ 0 & 0 & \mu^2 \end{bmatrix}$$

principal stretches: $\alpha\sqrt{2}$, $\beta\sqrt{2}$, μ

$$\underset{\sim}{U} = \underset{\sim}{C}^{1/2}$$

$$\underset{\sim}{F} = \underset{\sim}{R}\,\underset{\sim}{U}$$

$$\underset{\sim}{R} = \underset{\sim}{F}\,\underset{\sim}{U}^{-1} \quad \ldots\ldots$$

$$\underset{\sim}{R} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \qquad \underset{\sim}{U} = \begin{bmatrix} \alpha\sqrt{2} & 0 & 0 \\ 0 & \beta\sqrt{2} & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

c). $x_2^2 + x_3^2 = 4 \Leftrightarrow f(x_2, x_3) = 0$ $\qquad f(x_2, x_3) = x_2^2 + x_3^2 - 4$

$$\nabla f = (0, 2x_2, 2x_3)$$

$$\underset{(\text{lat. surface})}{\underline{n}} = \frac{\nabla f}{|\nabla f|} = \frac{1}{2}(0, x_2, x_3)$$

$$\underline{t}\Big|_{\text{lat. surface}} = \underset{\sim}{\sigma}\,\underline{n} = \frac{1}{2}\begin{bmatrix} 0 & -\alpha x_3 & \alpha x_2 \\ -\alpha x_3 & 0 & 0 \\ \alpha x_2 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix} = \underline{0}$$

end face $x_1 = 0$: $\underline{n} = (-1, 0, 0)$

$$\underline{t}\Big|_{x_1=0} = \begin{bmatrix} 0 & -\alpha x_3 & \alpha x_2 \\ -\alpha x_3 & 0 & 0 \\ \alpha x_2 & 0 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha x_3 \\ -\alpha x_2 \end{bmatrix}$$

$$\underline{t}\Big|_{x_1=L} = \begin{bmatrix} 0 & -\alpha x_3 & \alpha x_2 \\ -\alpha x_3 & 0 & 0 \\ \alpha x_2 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\alpha x_3 \\ \alpha x_2 \end{bmatrix}$$

end face $x_1 = L$
$\underline{n} = (+1, 0, 0)$

$$\underline{F}\Big|_{x_1=0} = \iint_{\substack{\text{cross}\\\text{section}}} \underline{t}\Big|_{x_1=0}\, da = \left(\iint_{\substack{\text{cross}\\\text{section}}} \alpha x_3\, da\right)\underline{e}_2 - \left(\iint_{\substack{\text{cross}\\\text{section}}} \alpha x_2\, da\right)\underline{e}_3$$

resultant on $x_1 = 0$

$da = r\,dr\,d\theta$ $\quad x_2 = r\cos\theta \quad x_3 = r\sin\theta \quad 0 \le r \le 2$, $\; 0 \le \theta < 2\pi$

$$\underline{F}\Big|_{x_1=0} = \alpha \int_0^{2\pi}\int_0^2 r^2 \sin\theta\, dr\, d\theta - \alpha \int_0^{2\pi}\int_0^2 r^2 \cos\theta\, d\theta = \underline{0}$$

Similarly, $\underline{F}\Big|_{x_1=L} = \underline{0}$

$$\underline{M}\Big|_{x_1=0} = \iint_{\substack{\text{cross}\\\text{section}}} \underline{x} \wedge \left(\underline{t}\Big|_{x_1=0}\right) da = -4\alpha\overbrace{\left(\iint_{\substack{\text{cross}\\\text{section}}} da\right)}^{\text{circle area}}\underline{e}_1 = -4\alpha(4\pi)\underline{e}_1$$
$$= \boxed{-16\pi\alpha\,\underline{e}_1}$$

resultant moment on $x_1 = 0$

$$\begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ x_1 & x_2 & x_3 \\ 0 & \alpha x_3 & -\alpha x_2 \end{vmatrix} = -\alpha(x_2^2 + x_3^2)\underline{e}_1 = -4\alpha\,\underline{e}_1$$

$$\underline{M}\Big|_{x_1=L} = +16\pi\alpha\,\underline{e}_1$$

Note that $\underline{M}\Big|_{x_1=0} + \underline{M}\Big|_{x_1=L} + \underline{M}\Big|_{\text{lat. surface}} = \underline{0}$

and $\underline{\mathcal{F}}\Big|_{x_1=0} + \underline{\mathcal{F}}\Big|_{x_1=L} + \underline{\mathcal{F}}\Big|_{\substack{\text{lat.}\\ \text{surface}}} = \underline{0}$

which confirm that our deformable body is in equilibrium.

The cylinder is twisted around the x_1- axis.

Q.4 b). $\mathcal{B}_R = \{\cancel{\underline{X}} \in \mathbb{R}^3 \mid X_1^2 + X_2^2 + X_3^2 \leq 1\}$

$\mathcal{B}_c = \underline{\chi}(\mathcal{B}_R) \Rightarrow$ deformed sphere

$$\text{vol}(\mathcal{B}_c) = \int_{\mathcal{B}_c} dv = \underbrace{\int_{\mathcal{B}_R} J\, dV}_{\text{must evaluate this integral}} \qquad J = \det \underset{\sim}{F}$$

$$\underset{\sim}{F} = \begin{bmatrix} 1 & \varepsilon X_3 & \varepsilon X_2 \\ \varepsilon X_3 & 1 & \varepsilon X_1 \\ \varepsilon X_2 & \varepsilon X_1 & 1 \end{bmatrix} \Rightarrow J = \det \underset{\sim}{F}$$
$$= 1 - \varepsilon^2 (X_1^2 + X_2^2 + X_3^2) + 2\varepsilon^3 X_1 X_2 X_3$$

To evaluate the integral $\rightarrow$ Mathematica

$\searrow$ or a simple change of variable (spherical coordinates)

$$(X_1, X_2, X_3) \longleftrightarrow (\rho, \theta, \varphi)$$

$$\begin{cases} X_1 = \rho \sin\varphi \cos\theta & 0 \leq \rho \leq 1 \\ X_2 = \rho \sin\varphi \sin\theta & 0 \leq \theta < 2\pi \\ X_3 = \rho \cos\varphi & 0 \leq \varphi \leq \pi \end{cases}$$

$$dV = \rho^2 \sin\varphi \, d\rho\, d\theta\, d\varphi$$

$$\text{vol}(\mathcal{B}_c) = \int_0^{\pi}\int_0^{2\pi}\int_0^{1} \rho^2 \sin\varphi \, d\rho\, d\theta\, d\rho - \varepsilon^2 \int_0^{\pi}\int_0^{2\pi}\int_0^{1} \rho^4 \sin\varphi\, d\rho\, d\theta\, d\varphi + \overbrace{\frac{2\varepsilon^3}{2}\int_0^{\pi}\int_0^{2\pi}\int_0^{1} \rho^5 \sin^3\varphi \cos\varphi \sin 2\theta \, d\rho\, d\theta\, d\varphi}^{0}$$

$$= \frac{2}{3}(2\pi) - \varepsilon^2\left(\frac{2}{5}\right)(2\pi) = \frac{4\pi}{3} - \frac{4\pi}{5}\varepsilon^2 = \frac{4\pi}{3}\left(1 - \frac{3}{5}\varepsilon^2\right)$$

c). $\underset{\sim}{A}$ second-order tensor

$$\det(\underset{\sim}{A} + \lambda \underset{\sim}{I}) = \lambda^3 + I_1\lambda^2 + I_2\lambda + I_3$$

$$I_1 = tr(\underset{\sim}{A}) \qquad I_2 = \frac{1}{2}\left[(tr(\underset{\sim}{A}))^2 - tr(\underset{\sim}{A}^2)\right] \qquad I_3 = \det \underset{\sim}{A}$$

Take $\underset{\sim}{A} \to \alpha \underset{\sim}{W}$ $\qquad \underset{\sim}{W}^T = -\underset{\sim}{W}$ $\qquad \lambda = 1$

$I_1 = 0 \qquad I_3 = 0 \qquad I_2 = \alpha^2$ (simple algebra)

$$\Rightarrow \det(\underset{\sim}{I} + \alpha\underset{\sim}{W}) = 1 + \alpha^2 > 0 \Rightarrow \underset{\sim}{I} + \alpha \underset{\sim}{W} \text{ exists}$$

Cayley-Hamilton:

$$(\underset{\sim}{I} + \alpha\underset{\sim}{W})^3 - I_1(\underset{\sim}{I} + \alpha\underset{\sim}{W})^2 + I_2(\underset{\sim}{I} + \alpha\underset{\sim}{W}) - I_3\underset{\sim}{I} = \underset{\sim}{0}$$

$$\Rightarrow (\underset{\sim}{I} + \alpha\underset{\sim}{W})^2 - I_1(\underset{\sim}{I} + \alpha\underset{\sim}{W}) + I_2\underset{\sim}{I} = I_3(\underset{\sim}{I} + \alpha\underset{\sim}{W})^{-1} \quad (*)$$

$$I_1 = tr(\underset{\sim}{I} + \alpha\underset{\sim}{W}) = tr(\underset{\sim}{I}) = 3$$

$$I_2 = \frac{1}{2}\left[(tr(\underset{\sim}{I} + \alpha\underset{\sim}{W}))^2 - tr(\underset{\sim}{I} + \alpha\underset{\sim}{W})(\underset{\sim}{I} + \alpha\underset{\sim}{W})\right]$$

$$= \frac{1}{2}\left[9 - tr(\underset{\sim}{I} + 2\alpha\underset{\sim}{W} + \alpha^2\underset{\sim}{W}^2)\right]$$

$$= \frac{1}{2}\left[9 - 3 - 2\alpha\cdot 0 + 2\alpha^2\right] = 3 + \alpha^2$$

$$I_3 = \det(\underset{\sim}{I} + \alpha\underset{\sim}{W}) = 1 + \alpha^2$$

$$\Rightarrow (1+\alpha^2)(\underset{\sim}{I}+\alpha\underset{\sim}{W})^{-1} = (\underset{\sim}{I}+\alpha\underset{\sim}{W})^2 - 3(\underset{\sim}{I}+\alpha\underset{\sim}{W}) + (3+\alpha^2)\underset{\sim}{I} \qquad (*)$$

From the solved examples:

$$\underset{\sim}{W}_1\underset{\sim}{W}_2 = \underline{\omega}_2 \otimes \underline{\omega}_1 - (\omega_1 \cdot \underline{\omega}_2)\underset{\sim}{I}$$

$$\underset{\sim}{W}_1 \to \underset{\sim}{W} \qquad \underset{\sim}{W}_2 \to \underset{\sim}{W} \qquad \underline{\omega}_1 = \underline{\omega}_2 \to \underline{\omega}$$

$$\Rightarrow \underset{\sim}{W}^2 = \underline{w}\otimes\underline{w} - \underbrace{|\underline{w}|^2}_{1}\underset{\sim}{I} = \underline{w}\otimes\underline{w} - \underset{\sim}{I}$$

$$\Rightarrow \underset{\sim}{I} + \underset{\sim}{W}^2 = \underline{w}\otimes\underline{w}$$

$$\underset{(*)}{\text{RHS}} = \underset{\sim}{I} + 2\alpha\underset{\sim}{W} + \alpha^2\underset{\sim}{W}^2 - 3\underset{\sim}{I} - 3\alpha\underset{\sim}{W} + (3+\alpha^2)\underset{\sim}{I}$$

$$= \alpha^2\underset{\sim}{W}^2 - \alpha\underset{\sim}{W} + (1+\alpha^2)\underset{\sim}{I}$$

$$= \alpha^2(\underset{\sim}{W}^2 + \underset{\sim}{I}) - \alpha\underset{\sim}{W} + \underset{\sim}{I}$$

$$= \alpha^2\,\underline{w}\otimes\underline{w} - \alpha\underset{\sim}{W} + \underset{\sim}{I}$$

Hence $(\underset{\sim}{I}+\alpha\underset{\sim}{W})^{-1} = (1+\alpha^2)^{-1}(\underset{\sim}{I} - \alpha\underset{\sim}{W} + \alpha^2\,\underline{w}\otimes\underline{w})$

$$\underset{\sim}{A} \to -\underset{\sim}{W} \qquad \underset{\sim}{B} \to \underline{w}\otimes\underline{w}$$

axial vector of $\underset{\sim}{W}$

a) $x_2^2 + 2x_3^2 = 1$

$f(x_2, x_3) = 0$ $\qquad f(x_2, x_3) = x_2^2 + 2x_3^2 - 1$

$$\nabla f = (0, 2x_2, 4x_3)$$

$$|\nabla f| = \sqrt{4x_2^2 + 16x_3^2} = \sqrt{4(1+4x_3^2)} = 2\sqrt{1+4x_3^2}$$

$$\underline{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{1+4x_3^2}}(0, x_2, 2x_3)$$

$$\underline{t}\Big|_{\substack{\text{lat}\\ \text{surface}}} = \underset{\sim}{\sigma}\,\underline{n} = \begin{bmatrix} 0 & -2x_3 & x_2 \\ -2x_3 & 0 & 0 \\ x_2 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ \frac{x_2}{\sqrt{1+4x_3^2}} \\ \frac{2x_3}{\sqrt{1+4x_3^2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \boxed{\underline{0}}$$

$$x_1 = 0 \Rightarrow \underline{n} = (-1, 0, 0)$$

$$\underline{t}\Big|_{x_1=0} = \begin{bmatrix} 0 & -2x_3 & x_2 \\ -2x_3 & 0 & 0 \\ x_2 & 0 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2x_3 \\ -x_2 \end{bmatrix}$$

$$\underline{F}\Big|_{x_1=0} = \left(\iint_{\substack{\text{cross}\\ \text{section}}} 2x_3\,da\right)\underline{e}_2 - \left(\iint_{\substack{\text{cross}\\ \text{section}}} x_2\,da\right)\underline{e}_3 = \boxed{\underline{0}}$$

resultant on $x_1 = 0$

cross section → elliptical domain $x_2^2 + 2x_3^2 \leq 1$

direct evaluation of the integrals

$$\underline{M}\Big|_{x_1=0} = \iint_{\substack{\text{cross}\\ \text{section}}} (\underline{x} \wedge \underline{t})\Big|_{x_1=0}\,da$$

resultant moment on $x_1=0$ w.r.t. O

$$\begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ 0 & x_2 & x_3 \\ 0 & 2x_3 & -x_2 \end{vmatrix} = (-x_2^2 - 2x_3^2)\,\underline{e}_1$$

Since $\iint_{\substack{\text{elliptical}\\ \text{domain}}} x_2^2\,da = \frac{\pi}{4\sqrt{2}}$ and $\iint_{\substack{\text{elliptical}\\ \text{domain}}} x_3^2\,da = \frac{\pi}{8\sqrt{2}}$

$$\hookrightarrow \boxed{\underline{M}\Big|_{x_1=0} = -\frac{\pi}{2\sqrt{2}}\,\underline{e}_1}$$

Q.5 a) $\underline{V}(\underline{X},t) = \frac{\partial \underline{\chi}}{\partial t}(\underline{X},t) = (3t^2 X_3, 3t^2 X_1, 3t^2 X_2)$

$\Rightarrow \underline{V}(\underline{X}^\circ, t) = (12t^2, 12t^2, 12t^2)$

$\underline{v}(\underline{x}^\circ, t) = \underline{V}(\underline{\chi}^{-1}(\underline{x}^\circ, t), t)$

$$\begin{bmatrix} 1 & 0 & t^3 \\ t^3 & 1 & 0 \\ 0 & t^3 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \text{Cramer's Rule}$$

$$X_1 = \frac{\begin{vmatrix} x_1 & 0 & t^3 \\ x_2 & 1 & 0 \\ x_3 & t^3 & 1 \end{vmatrix}}{1+t^9} \qquad X_2 = \frac{\begin{vmatrix} 1 & x_1 & t^3 \\ t^3 & x_2 & 0 \\ 0 & x_3 & 1 \end{vmatrix}}{1+t^9} \qquad X_3 = \frac{\begin{vmatrix} 1 & 0 & x_1 \\ t^3 & 1 & x_2 \\ 0 & t^3 & x_3 \end{vmatrix}}{1+t^9}$$

this is $\underline{\chi}^{-1}(\underline{x}, t)$

$$\underline{\chi}^{-1}(\underline{x}^\circ, t) = \left(\frac{4(t^6 - t^3 + 1)}{1+t^9}, \frac{4(t^6 - t^3 + 1)}{1+t^9}, \frac{4(t^6 - t^3 + 1)}{1+t^9} \right)$$

$$v_1 = v_2 = v_3 = \frac{12t^2 \cancel{(t^6 - t^3 + 1)}}{(t^3+1)\cancel{(t^6 - t^3 + 1)}} = \frac{12t^2}{t^3+1}$$

b) Let $\underline{n}, \underline{p}$ be unit vectors s.t. $\{\underline{m}, \underline{n}, \underline{p}\}$ is an orthonormal system

Then

$$\underset{\sim}{I} = \underline{m} \otimes \underline{m} + \underline{n} \otimes \underline{n} + \underline{p} \otimes \underline{p}$$

$$\Rightarrow \underset{\sim}{A} = \alpha\, \underline{m} \otimes \underline{m} + \alpha\, \underline{n} \otimes \underline{n} + \alpha\, \underline{p} \otimes \underline{p} + \beta\, \underline{m} \otimes \underline{m}$$

$$= (\alpha + \beta)\, \underline{m} \otimes \underline{m} + \alpha (\underline{n} \otimes \underline{n} + \underline{p} \otimes \underline{p})$$

The eigenvalues of $\underset{\sim}{A}$ are $\alpha + \beta$, α, α, and the spectral representation is

$$\underset{\sim}{A} = (\alpha + \beta)\, \underline{m} \otimes \underline{m} + \alpha (\underset{\sim}{I} - \underline{m} \otimes \underline{m})$$

$\dot{\underline{x}}$ is objective $\iff$ $\underset{\sim}{Q} = \text{const.}$ and $\underline{c} = \text{const.}$ (i.e. no time dependence)

$$\dot{\underline{x}}^* - \underset{\sim}{Q}\dot{\underline{x}} = \underset{\sim}{\Omega}_Q(\underline{x}^* - \underline{c}) + \dot{\underline{c}} \qquad \underset{\sim}{\Omega}_Q := \dot{\underset{\sim}{Q}}\underset{\sim}{Q}^T$$

by re-arranging (*)

Differentiating with respect to t and separating the non-objective terms as above

$$\Rightarrow \ddot{\underline{x}}^* - \underset{\sim}{Q}\ddot{\underline{x}} = \dot{\underset{\sim}{\Omega}}_Q(\underline{x}^* - \underline{c}) - \underset{\sim}{\Omega}_Q^2(\underline{x}^* - \underline{c}) + 2\underset{\sim}{\Omega}_Q(\dot{\underline{x}} - \dot{\underline{c}}) + \ddot{\underline{c}}$$

The acceleration is objective provided that

$$\dot{\underline{c}} = \text{const} \quad \text{and} \quad \underset{\sim}{Q} = \text{const.}$$

$\downarrow$

translational motion with constant velocity

d). $E = \dfrac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$ $\qquad \nu = \dfrac{\lambda}{2(\lambda+\mu)}$ $\qquad$ [Lecture #9]

$$\frac{1}{1-2\nu} = \frac{1}{1 - \frac{2\lambda}{2(\lambda+\mu)}} = \frac{\lambda+\mu}{\mu}$$ so the equation is the same as (4)

$$\nabla^2\underline{u} + \frac{1}{1-2\nu}\,\text{grad}(\text{div}\,\underline{u})$$

$$= \nabla^2\underline{\Psi} + \frac{1}{1-2\nu}\,\text{grad}(\text{div}\,\underline{\Psi}) - \frac{1}{4(1-\nu)}\Big[\nabla^2\,\text{grad}(\underline{x}\cdot\underline{\Psi} + \Phi) + \frac{1}{1-2\nu}\,\text{grad}\,\nabla^2(\underline{x}\cdot\underline{\Psi} + \Phi)\Big]$$

For the other tensor:

$$\underset{\sim}{B}(\underline{m}+\underline{n}) = (\underline{m}\otimes\underline{n}+\underline{n}\otimes\underline{m})(\underline{m}+\underline{n}) = \dots = \underline{m}+\underline{n}$$

$$\underset{\sim}{B}(\underline{m}-\underline{n}) = \dots = (-1)(\underline{m}-\underline{n})$$

$$\underset{\sim}{B}(\underline{m}\wedge\underline{n}) = 0\,(\underline{m}\wedge\underline{n})$$

Hence the eigenvalues of $\underset{\sim}{B}$ are $+1, -1, 0$. The corresponding eigenvec.

$$\underline{m}+\underline{n}, \quad \underline{m}-\underline{n}, \quad \underline{m}\wedge\underline{n}$$

Note that $|\underline{m}+\underline{n}|^2 = (\underline{m}+\underline{n})\cdot(\underline{m}+\underline{n}) = \dots = 2$

Normalized eigenvectors

$$\frac{1}{\sqrt{2}}(\underline{m}+\underline{n}), \quad \frac{1}{\sqrt{2}}(\underline{m}-\underline{n}), \quad \underline{m}\wedge\underline{n}$$

Spectral representation:

$$\underset{\sim}{B} = \frac{1}{2}(\underline{m}+\underline{n})\otimes(\underline{m}+\underline{n}) - \frac{1}{2}(\underline{m}-\underline{n})\otimes(\underline{m}-\underline{n})$$

c). $\underset{\sim}{\sigma}$ objective (notes)

$\Rightarrow$ div $\underset{\sim}{\sigma}$ objective (Prop. 3.1 / notes)

$\rho \underline{b}$ objective

In order for the equation to be objective $\Rightarrow$ $\underline{a}$ must be objective

$$\underline{x}^* = \underset{\sim}{Q}\underline{x} + \underline{c} \qquad \underset{\sim}{Q} = \underset{\sim}{Q}(t) \qquad \underline{c} = \underline{c}(t)$$

$$\underset{\sim}{Q}^T\underset{\sim}{Q} = \underset{\sim}{Q}\underset{\sim}{Q}^T = \underset{\sim}{I}$$

$$\Rightarrow \dot{\underline{x}}^* = \dot{\underset{\sim}{Q}}\underline{x} + \underset{\sim}{Q}\dot{\underline{x}} + \dot{\underline{c}}$$

$$\Rightarrow \dot{\underline{x}}^* - \underset{\sim}{Q}\dot{\underline{x}} = \dot{\underset{\sim}{Q}}\underline{x} + \dot{\underline{c}} \quad (*)$$

Since $\nabla^2(\text{grad}) = \text{grad}(\nabla^2)$ and (easy to check)

$$\nabla^2(\underline{x}\cdot\underline{\Psi}) = \underline{x}\cdot(\nabla^2\underline{\Psi}) + 2\,\text{div}\,\underline{\Psi}$$

$$\Rightarrow \nabla^2\underline{u} + \frac{1}{1-2\nu}\,\text{grad}(\text{div}\,\underline{u})$$

$$= \nabla^2\underline{\Psi} + \frac{1}{1-2\nu}\,\text{grad}(\text{div}\,\underline{\Psi}) - \frac{1}{2(1-2\nu)}\,\text{grad}\left(\overbrace{\underline{x}\cdot\nabla^2\underline{\Psi} + \nabla^2\phi}^{0\,=} + 2\,\text{div}\,\underline{\Psi}\right)$$

But $\underline{x}\cdot\nabla^2\underline{\Psi} = -\nabla^2\phi$ and $\nabla^2\underline{\Psi} = -\frac{1}{\mu}\underline{f}$

$$\Rightarrow \nabla^2\underline{u} + \frac{1}{1-2\nu}\,\text{grad}(\text{div}\,\underline{u}) = -\frac{1}{\mu}\underline{f} \Rightarrow \text{QED.}$$