Intro to Continuum Mechanics 2018
(solutions for the coursework)

Q.1 a) Let $\underset{\sim}{T} := (\underline{a}\otimes\underline{b} + \underline{c}\otimes\underline{d})^*$

$\underset{\sim}{S} := (\underline{a}\wedge\underline{c})\otimes(\underline{b}\wedge\underline{d})$

Must show that $\underset{\sim}{T}(\underline{u}\wedge\underline{v}) = \underset{\sim}{S}(\underline{u}\wedge\underline{v}) \quad (\forall)\ \underline{u},\underline{v}\in\mathcal{V}$

$$\text{LHS} = (\underline{a}\otimes\underline{b}+\underline{c}\otimes\underline{d})^*(\underline{u}\wedge\underline{v}) \underset{\text{def of }*}{=} [(\underline{a}\otimes\underline{b}+\underline{c}\otimes\underline{d})\underline{u}]\wedge[(\underline{a}\otimes\underline{b}+\underline{c}\otimes\underline{d})\underline{v}]$$

$$= [\underline{a}(\underline{b}\cdot\underline{u})+\underline{c}(\underline{d}\cdot\underline{u})]\wedge[\underline{a}(\underline{b}\cdot\underline{v})+\underline{c}(\underline{d}\cdot\underline{v})]$$

$$= (\underline{a}\wedge\underline{c})(\underline{b}\cdot\underline{u})(\underline{d}\cdot\underline{v}) + (\underline{c}\wedge\underline{a})(\underline{d}\cdot\underline{u})(\underline{b}\cdot\underline{v})$$

$$= (\underline{a}\wedge\underline{c})[(\underline{b}\cdot\underline{u})(\underline{d}\cdot\underline{v}) - (\underline{d}\cdot\underline{u})(\underline{b}\cdot\underline{v})]$$

$$\overset{(I)}{=} (\underline{a}\wedge\underline{c})[(\underline{b}\wedge\underline{d})\cdot(\underline{u}\wedge\underline{v})] \qquad (1)$$

$$\text{RHS} = [(\underline{a}\wedge\underline{c})\otimes(\underline{b}\wedge\underline{d})](\underline{u}\wedge\underline{v}) = (\underline{a}\wedge\underline{c})[(\underline{b}\wedge\underline{d})\cdot(\underline{u}\wedge\underline{v})] \qquad (2)$$

OBS. In (I) I used the formula

$$(\underline{A}\wedge\underline{B})\cdot(\underline{C}\wedge\underline{D}) = \begin{vmatrix} \underline{A}\cdot\underline{C} & \underline{A}\cdot\underline{D} \\ \underline{B}\cdot\underline{C} & \underline{B}\cdot\underline{D} \end{vmatrix} \qquad \underline{A},\underline{B},\underline{C},\underline{D} = \text{vectors}$$

Comparing (1) and (2) $\Rightarrow$ QED

b). If $\underline{a},\underline{b},\underline{c}$ are orthonormal $\Rightarrow$ $\underline{a}\wedge\underline{b}=\underline{c}$, $\underline{b}\wedge\underline{c}=\underline{a}$, $\underline{c}\wedge\underline{a}=\underline{b}$
and these vectors have unit magnitude

Take $\underline{d}=\underline{c}$ in the identity proved above

$$(\underline{a}\otimes\underline{b}+\underline{c}\otimes\underline{c})^* = \underbrace{(\underline{a}\wedge\underline{c})}_{-\underline{b}}\otimes\underbrace{(\underline{b}\wedge\underline{c})}_{\underline{a}} \Rightarrow \underline{b}\otimes\underline{a} = -(\underline{a}\otimes\underline{b}+\underline{c}\otimes\underline{c})^*$$

QED

b). Let $\underline{v}, \underline{w}$ be two unit vectors such that $\{\underline{u}, \underline{v}, \underline{w}\}$ forms an orthonormal system

Then $\underset{\sim}{I} = \underline{u}\otimes\underline{u} + \underline{v}\otimes\underline{v} + \underline{w}\otimes\underline{w}$

$$\Rightarrow \underset{\sim}{I} - \underline{u}\otimes\underline{u} = \underline{v}\otimes\underline{v} + \underline{w}\otimes\underline{w}$$

(1) in Q1

$$\Rightarrow (\underset{\sim}{I} - \underline{u}\otimes\underline{u})^* = (\underline{v}\otimes\underline{v} + \underline{w}\otimes\underline{w})^* = (\underline{v}\wedge\underline{w})\otimes(\underline{v}\wedge\underline{w})$$

$$= \underline{u}\otimes\underline{u}$$

If $|\underline{u}| \neq 1$ then let $\hat{\underline{u}} := \frac{\underline{u}}{|\underline{u}|}$ which is a unit vector

Consider again $\underline{v}, \underline{w}$ s.t. $\{\hat{\underline{u}}, \underline{v}, \underline{w}\}$ is an orthonormal system

$$\underset{\sim}{I} - \underline{u}\otimes\underline{u} = \hat{\underline{u}}\otimes\hat{\underline{u}} + \underline{v}\otimes\underline{v} + \underline{w}\otimes\underline{w} - |\underline{u}|^2\,\hat{\underline{u}}\otimes\hat{\underline{u}}$$

$$= (1-|\underline{u}|^2)\,\hat{\underline{u}}\otimes\hat{\underline{u}} + \underline{v}\otimes\underline{v} + \underline{w}\otimes\underline{w} \qquad (*)$$

[OBS. if $\underset{\sim}{A} \in CT(2) \Rightarrow \underset{\sim}{A}^* = (\det \underset{\sim}{A})\,\underset{\sim}{A}^{-T}$]

Let $\underset{\sim}{A}$ be the RHS of $(*)$.

$$\det \underset{\sim}{A} = (1-|\underline{u}|^2)\cdot 1\cdot 1 = 1-|\underline{u}|^2$$

$$\underset{\sim}{A}^T = \underset{\sim}{A} \quad \text{and} \quad \underset{\sim}{A}^{-1} = \frac{1}{1-|\underline{u}|^2}\,\hat{\underline{u}}\otimes\hat{\underline{u}} + \underline{v}\otimes\underline{v} + \underline{w}\otimes\underline{w}$$

$$(\underset{\sim}{I} - \underline{u}\otimes\underline{u})^* = (1-|\underline{u}|^2)\left[\frac{1}{1-|\underline{u}|^2}\,\hat{\underline{u}}\otimes\hat{\underline{u}} + \underline{v}\otimes\underline{v} + \underline{w}\otimes\underline{w}\right]$$