# LECTURE 10

The solution of BVP in linear elasticity is based on a number of further simplifications:

1. **Principle of superposition** → one can break up a problem in two or more simpler problems; the solution of the original problem will be the sum of the solutions of the individual sub-problems.

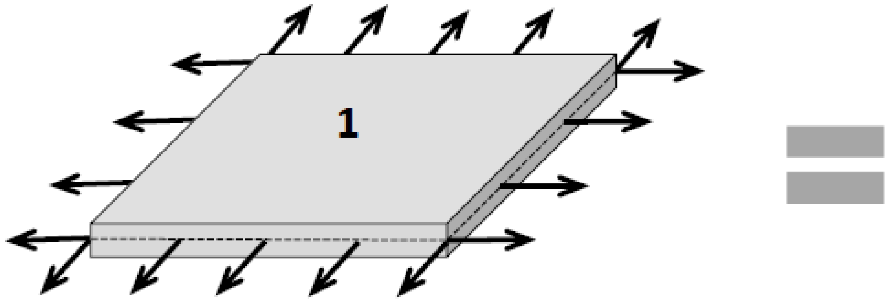
1. **St. Venant's Principle** (for stresses)→ allows us to simplify BCs.

Solution strategy (to get closed-form solutions) ⟶ **semi-inverse method**

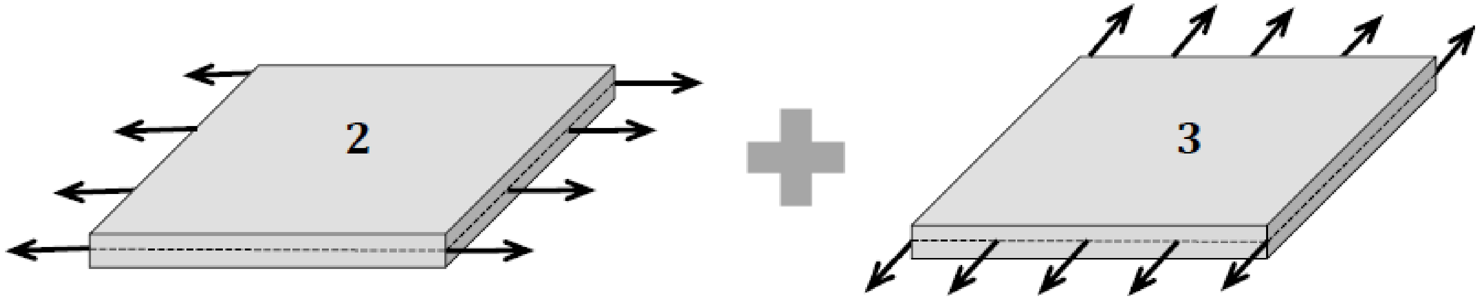
Some assumptions are made about the components of stress, the components of strain or the components of the displacement, while leaving sufficient freedom in these assumptions so that the equations of linear elasticity may be satisfied. If these assumptions allow us to satisfy all the equations for the BVP in question, then, by the **uniqueness theorem,** we have succeeded in obtaining the solution to the problem.

# PRINCIPLE OF SUPERPOSITION

$$\{\boldsymbol{u}^{(1)}, \boldsymbol{E}^{(1)}, \boldsymbol{T}^{(1)}\}$$




$$\{\boldsymbol{u}^{(j)}, \boldsymbol{E}^{(j)}, \boldsymbol{T}^{(j)}\}\ (j = 2, 3)$$




$$\boldsymbol{u}^{(1)} = \boldsymbol{u}^{(2)} + \boldsymbol{u}^{(3)}, \qquad \boldsymbol{E}^{(1)} = \boldsymbol{E}^{(2)} + \boldsymbol{E}^{(3)}, \qquad \boldsymbol{T}^{(1)} = \boldsymbol{T}^{(2)} + \boldsymbol{T}^{(3)}$$

# PRINCIPLE OF SUPERPOSITION

**Proposition 2.2.** *(The Superposition Principle)* *Let $\mathcal{B}$ be an elastic body occupying a domain $\Omega_0 \subset \mathbb{R}^3$ in the reference configuration. Consider the elastic states $\{\boldsymbol{u}^{(j)}, \boldsymbol{E}^{(j)}, \boldsymbol{T}^{(j)}\}$ $(j = 1,2,\ldots,m)$ corresponding to the elastostatic problem defined by the differential equations (2.37b)-(2.37c), (2.38), with body force $\mathsf{b} \to \mathsf{b}^{(j)}$, and subject to*

$$\boldsymbol{u}^{(j)}\Big|_{\partial\Omega_0^u} = \mathring{\boldsymbol{u}}^{(j)} \qquad \text{and} \qquad [\boldsymbol{n}\cdot\boldsymbol{T}^{(j)}]\Big|_{\partial\Omega_0^t} = \mathring{\boldsymbol{t}}^{(j)}, \qquad (j = 1,2,\ldots,m).$$

$$\nabla \cdot \boldsymbol{T} + \rho \boldsymbol{b} = 0;$$

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{u} \otimes \nabla + \nabla \otimes \boldsymbol{u});$$

$$\boldsymbol{T} = 2\mu \boldsymbol{E} + \lambda |\boldsymbol{E}| \boldsymbol{I},$$

*Then the elastic state $\{\boldsymbol{u}, \boldsymbol{E}, \boldsymbol{T}\}$, where*

$$\boldsymbol{u} \equiv \sum_{j=1}^{m} \boldsymbol{u}^{(j)}, \qquad \boldsymbol{E} \equiv \sum_{j=1}^{m} \boldsymbol{E}^{(j)}, \qquad \boldsymbol{T} \equiv \sum_{j=1}^{m} \boldsymbol{T}^{(j)}$$

*is the solution of the same differential system, with $\mathsf{b} \to \sum_{j=1}^{m} \mathsf{b}^{(j)}$ and the modified boundary conditions*

$$\boldsymbol{u}\Big|_{\partial\Omega_0^u} = \sum_{j=1}^{m} \mathring{\boldsymbol{u}}^{(j)} \qquad \text{and} \qquad (\boldsymbol{n}\cdot\boldsymbol{T})\Big|_{\partial\Omega_0^t} = \sum_{j=1}^{m} \mathring{\boldsymbol{t}}^{(j)}, \qquad (j = 1,2,\ldots,m).$$

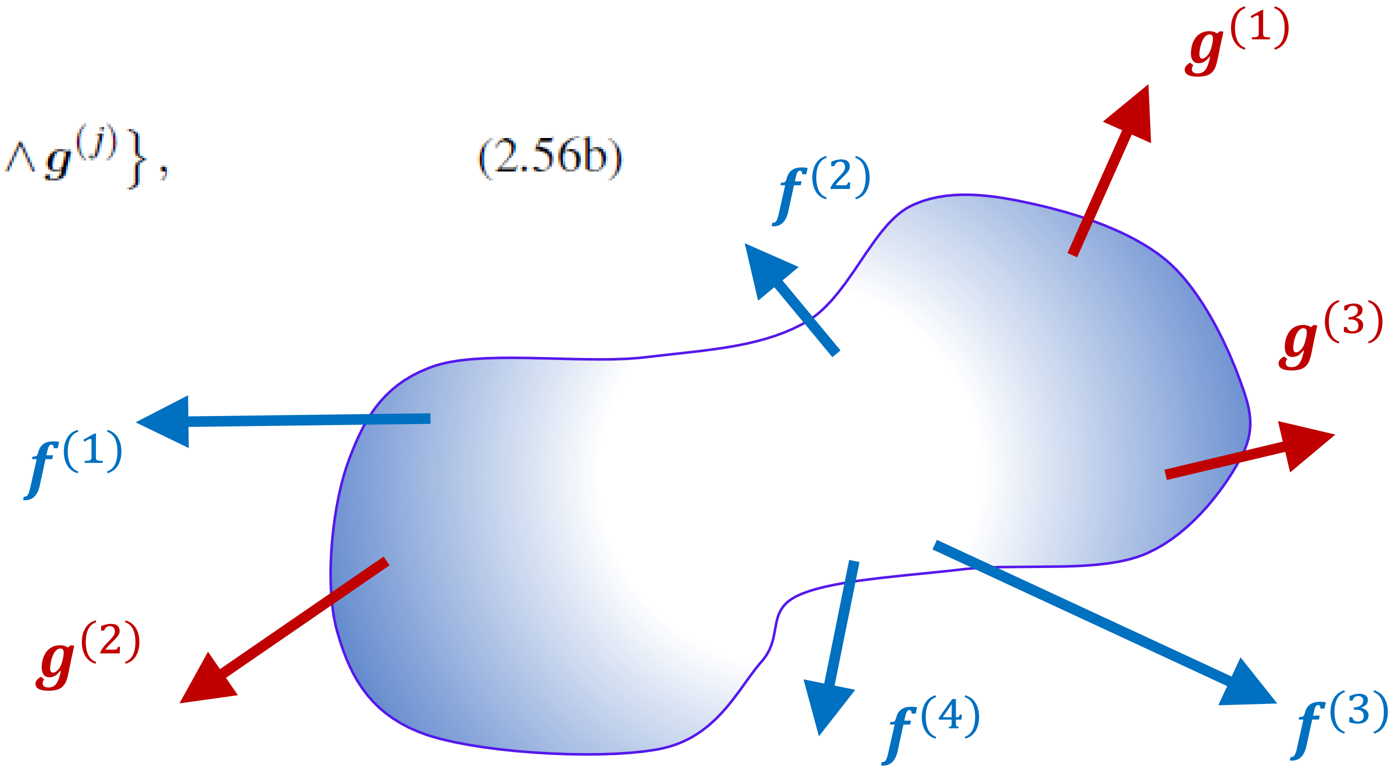
# ST. VENANT'S PRINCIPLE

**Definition 2.1.** Let $\mathcal{B}$ be an elastic body. Consider two systems of forces $\{\boldsymbol{f}^{(i)}\}_{i\in I}$ and $\{\boldsymbol{g}^{(j)}\}_{j\in J}$ acting at the points $O_i \in \mathcal{B}$ $(i \in I)$ and $O_j{}' \in \mathcal{B}$ $(j \in J)$, respectively. These systems of forces are said to be *statically equivalent* (or *equipollent*) if the following two conditions are satisfied

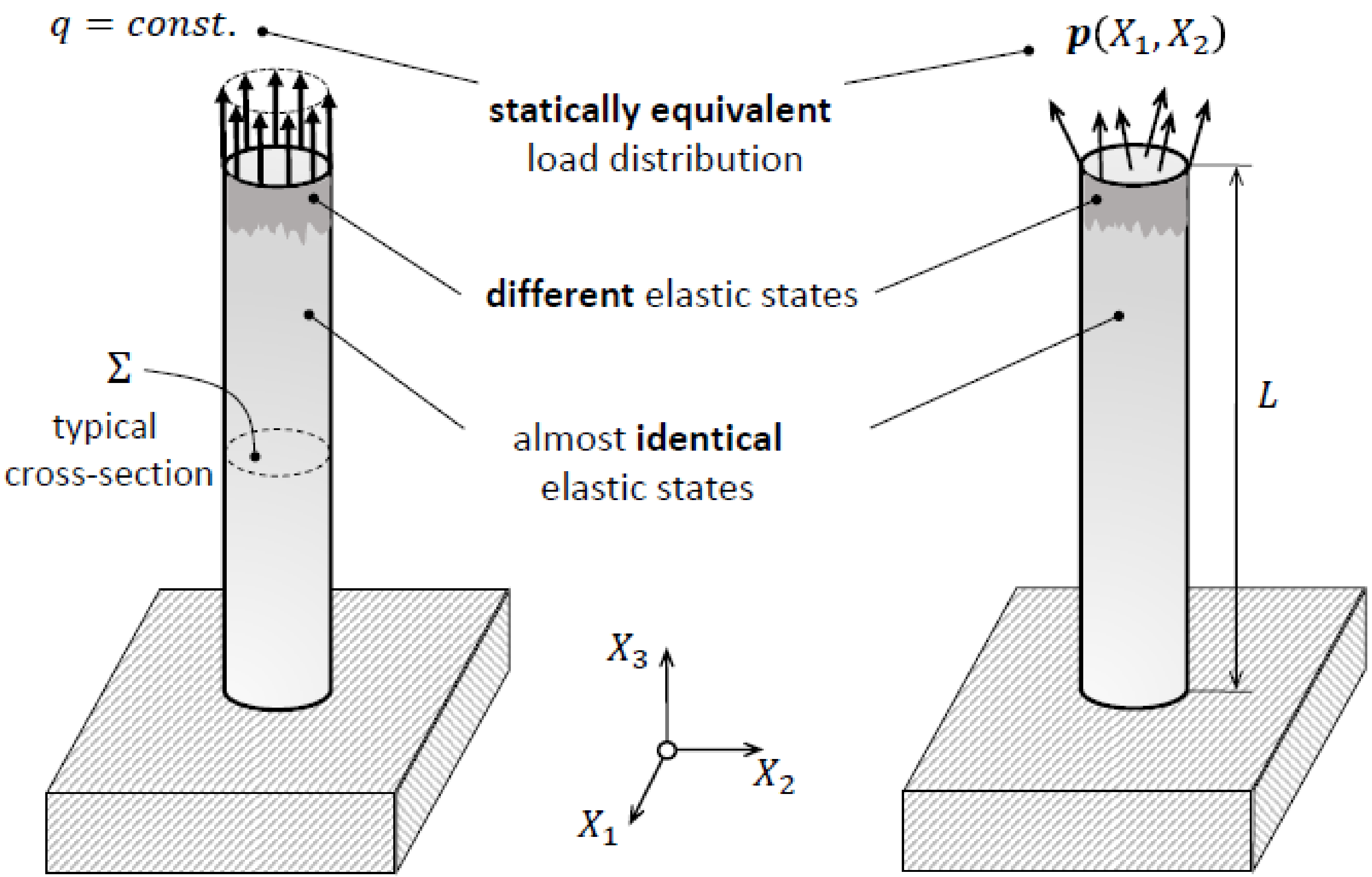
$$\sum_{i\in I} \boldsymbol{f}^{(i)} = \sum_{j\in J} \boldsymbol{g}^{(j)}, \tag{2.56a}$$

$$\sum_{i\in I} \{\overrightarrow{OO_i} \wedge \boldsymbol{f}^{(i)}\} = \sum_{j\in J} \{\overrightarrow{OO_j{}'} \wedge \boldsymbol{g}^{(j)}\}, \tag{2.56b}$$

for some point $O \in \mathcal{B}$.



# ST. VENANT'S PRINCIPLE
(intuitive illustration)



# ST. VENANT'S PRINCIPLE

**Proposition 2.3.** *(St. Venant's Principle)* *For an elastic body $\mathcal{B}$ occupying a region $\Omega_0 \subset \mathbb{R}^3$ consider the two separate boundary-value problems of linear elastostatics consisting of the differential equations (2.37b)-(2.37c) and (2.38) subject to*

$$\boldsymbol{u}\Big|_{\partial\Omega_0^u} = \mathring{\boldsymbol{u}} \qquad \text{and} \qquad (\boldsymbol{n}\cdot\boldsymbol{T})\Big|_{\partial\Omega_0^t} = \mathring{\boldsymbol{t}}^{(j)}, \qquad (j = 1, 2).$$

*where $\mathring{\boldsymbol{t}}^{(1)} \equiv \mathring{\boldsymbol{t}}^{(1)}(\boldsymbol{X})$ and $\mathring{\boldsymbol{t}}^{(2)} \equiv \mathring{\boldsymbol{t}}^{(2)}(\boldsymbol{X})$, $(\boldsymbol{X} \in \partial\Omega_0^t)$ are statically equivalent traction distributions, i.e.*

$$\int_{\partial\Omega_0^t} \mathring{\boldsymbol{t}}^{(1)}(\boldsymbol{X})\,\mathrm{d}A = \int_{\partial\Omega_0^t} \mathring{\boldsymbol{t}}^{(2)}(\boldsymbol{X})\,\mathrm{d}A, \tag{2.57a}$$

$$\int_{\partial\Omega_0^t} \boldsymbol{X} \wedge \mathring{\boldsymbol{t}}^{(1)}(\boldsymbol{X})\,\mathrm{d}A = \int_{\partial\Omega_0^t} \boldsymbol{X} \wedge \mathring{\boldsymbol{t}}^{(2)}(\boldsymbol{X})\,\mathrm{d}A. \tag{2.57b}$$

*If area $(\partial\Omega_0^t) \ll$ area $(\partial\Omega_0)$ then*

$$\{\boldsymbol{u}^{(1)}, \boldsymbol{E}^{(1)}, \boldsymbol{T}^{(1)}\} \simeq \{\boldsymbol{u}^{(2)}, \boldsymbol{E}^{(2)}, \boldsymbol{T}^{(2)}\}$$

*at points in $\Omega_0$ away from $\partial\Omega_0^t$. In the vicinity of $\partial\Omega_0^t$ the two elastic states are in general different.*

$$\boldsymbol{\nabla}\cdot\boldsymbol{T} + \rho\boldsymbol{b} = \boldsymbol{0};$$

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{u}\otimes\boldsymbol{\nabla} + \boldsymbol{\nabla}\otimes\boldsymbol{u});$$

$$\boldsymbol{T} = 2\mu\boldsymbol{E} + \lambda|\boldsymbol{E}|\boldsymbol{I},$$

**ASIDE:**

$A(x,y), B(x,y) =$ given continuously differentiable functions

$? f = f(x,y)$ such that $\dfrac{\partial f}{\partial x} = A(x,y)$ and $\dfrac{\partial f}{\partial y} = B(x,y)$

this system of simultaneous PDEs is **solvable** $\Longleftrightarrow$ $$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$$

**compatibility** condition for our PDEs

# COMPATIBILITY IN LINEAR ELASTICITY

$$\frac{\partial u_1}{\partial X_1} = E_{11}\,, \qquad \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right) = E_{12}$$

$$\frac{\partial u_2}{\partial X_2} = E_{22}\,, \qquad \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\right) = E_{23}$$

$$\frac{\partial u_3}{\partial X_3} = E_{33}\,, \qquad \frac{1}{2}\left(\frac{\partial u_3}{\partial X_1} + \frac{\partial u_1}{\partial X_3}\right) = E_{31}$$

$$\boxed{\begin{aligned}
2E_{12,12} &= E_{11,22} + E_{22,11}\\
2E_{13,13} &= E_{11,33} + E_{33,11}\\
2E_{23,23} &= E_{33,22} + E_{22,33}\\
E_{11,23} &= (E_{12,3} - E_{23,1} + E_{31,2})_{,1}\\
E_{22,13} &= (E_{21,3} - E_{13,2} + E_{32,1})_{,2}\\
E_{33,12} &= (E_{31,2} - E_{12,3} + E_{23,1})_{,3}
\end{aligned}}$$

Since $\boldsymbol{E}$ is related to $\boldsymbol{T}$, the **compatibility conditions** for the strain tensor can be shown to be equivalent to a (tensorial) equation for the stress tensor $T$.

**Beltrami-Michell equations:**

$$\nabla^2 T + \frac{1}{1+\nu}\boldsymbol{\nabla}\otimes(\boldsymbol{\nabla}|T|) = -\frac{\nu}{1-\nu}(\boldsymbol{\nabla}\cdot f)\boldsymbol{I} - (\boldsymbol{\nabla}\otimes f + f\otimes\boldsymbol{\nabla})$$

$tr(\boldsymbol{T})$

$\boldsymbol{f} := \rho\boldsymbol{b}$

The governing equations of linear elasticity can be also formulated in terms of the displacement field.

**Navier-Lame system:**

$$\mu\nabla^2 \boldsymbol{u} + (\lambda+\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) + \rho\boldsymbol{b} = \rho\frac{\partial^2 \boldsymbol{u}}{\partial t^2}$$

(see the solved examples for Chapter 1)

$$\boldsymbol{t} \equiv \boldsymbol{t}(\boldsymbol{u}) = \lambda(\boldsymbol{\nabla}\cdot\boldsymbol{u})\boldsymbol{n} + 2\mu(\boldsymbol{n}\cdot\boldsymbol{\nabla})\boldsymbol{u} + \mu(\boldsymbol{n}\wedge(\boldsymbol{\nabla}\wedge\boldsymbol{u}))$$

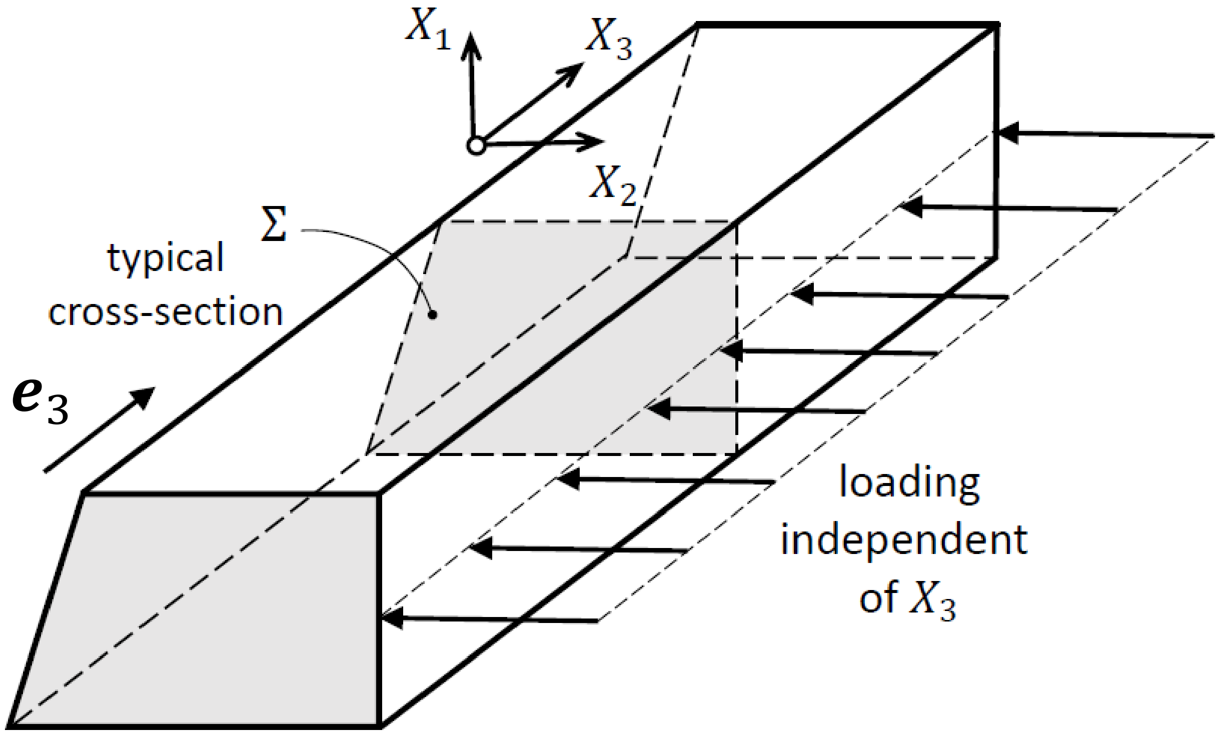
traction vector expressed
In terms of the displacement field
(to use in the stress BCs)

# PLANE STRAIN

$$u_3 \equiv 0, \qquad u_\alpha = u_\alpha(X_1, X_2), \qquad \alpha = 1,2$$

$$[\boldsymbol{E}] = \begin{bmatrix} E_{11} & E_{12} & 0 \\ E_{21} & E_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_{\alpha\beta} = E_{\alpha\beta}(X_1, X_2) \text{ for } \alpha, \beta = 1,2$$




$\boldsymbol{E}_{int}$

$$\begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix}$$

$\boldsymbol{T}_{int}$

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{12} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33} \end{bmatrix}$$

We use Hooke's law to write a reduced form of the constitutive eqn. for the interior parts of $\boldsymbol{E}$ and $\boldsymbol{T}$

$$\boldsymbol{T}_{\text{int}} = 2\mu \boldsymbol{E}_{\text{int}} + \lambda |\boldsymbol{E}_{\text{int}}| \boldsymbol{I}_2$$

$$\boldsymbol{I}_2 = \boldsymbol{e}_\alpha \otimes \boldsymbol{e}_\alpha$$

Greek subscripts
range from 1 to 2

$$T_{33} = \lambda (E_{\gamma\gamma})$$

$$T_{\alpha\alpha} = 2(\lambda + \mu) E_{\gamma\gamma}$$

$$T_{33} = \frac{\lambda}{2(\lambda + \mu)} T_{\gamma\gamma} \implies \boxed{T_{33} = \nu (T_{11} + T_{22})}$$

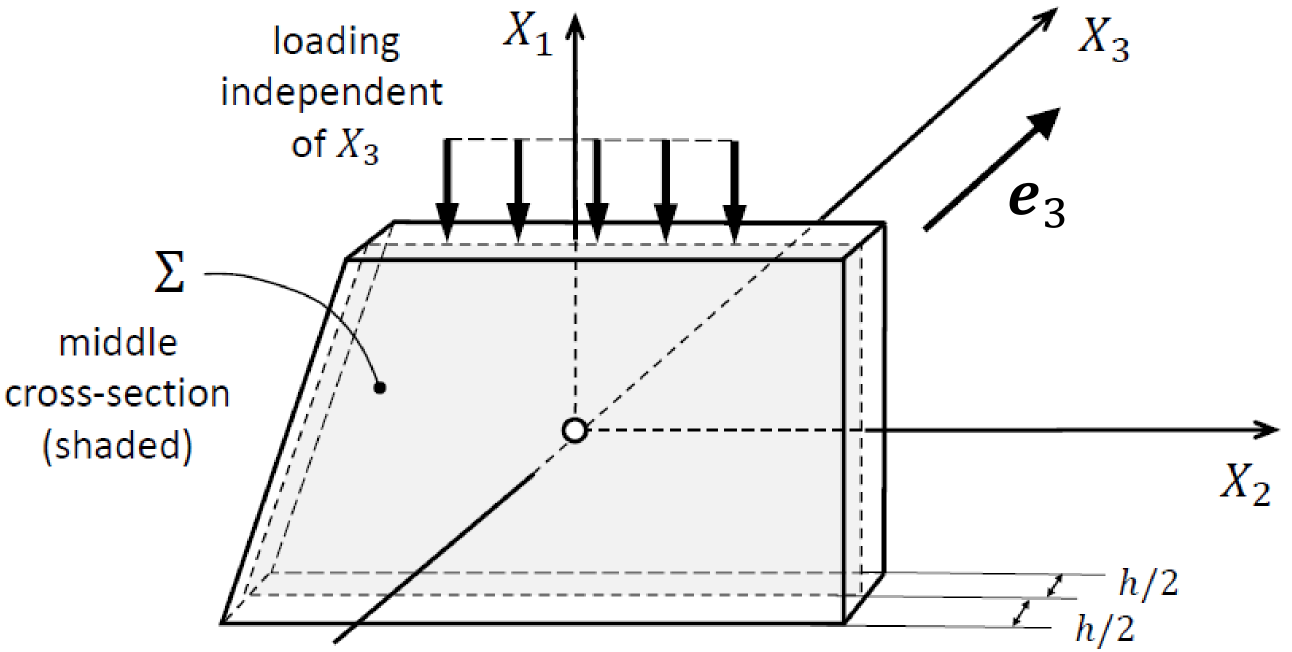
- this represents the stress component perpendicular to Σ
- it is needed for the consistency of our approximation
- plays no role in the actual solution of the **plane-strain** problem

# PLANE STRESS

$$T_{i3} = 0, \quad (i = 1,2,3)$$

$$[T] = \begin{bmatrix} T_{11} & T_{12} & 0 \\ T_{21} & T_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$T_{\alpha\beta} = T_{\alpha\beta}(X_1, X_2), \quad (\alpha, \beta = 1,2)$$




$$\begin{array}{c} \boldsymbol{E}_{int} \\ \downarrow \\ \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \end{array} \qquad \begin{array}{c} \boldsymbol{T}_{int} \\ \downarrow \\ \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{12} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33} \end{bmatrix} \end{array}$$

$$\boldsymbol{E}_{\text{int}} = \frac{1+\nu}{E}\boldsymbol{T}_{\text{int}} - \frac{\nu}{E}|\boldsymbol{T}_{\text{int}}|\boldsymbol{I}_2$$

$$E_{33} = -(\nu/E)(T_{\gamma\gamma})$$

$$E_{33} = -\frac{\nu}{1-\nu}E_{\alpha\alpha} \quad \Longrightarrow \quad E_{\gamma\gamma} + E_{33} = \frac{2\mu}{\lambda+2\mu}E_{\gamma\gamma}$$

$$\boldsymbol{T} = 2\mu\boldsymbol{E} + \lambda|\boldsymbol{E}|\boldsymbol{I} \quad \Rightarrow \quad \boxed{\boldsymbol{T}_{\text{int}} = 2\mu\boldsymbol{E}_{\text{int}} + \frac{2\lambda\mu}{\lambda+2\mu}|\boldsymbol{E}_{\text{int}}|\boldsymbol{I}_2}$$

Hooke’s law for plane-stress elasticity

# GOVERNING EQUATIONS FOR PLANE ELASTICITY

## CARTESIAN

$$\frac{\partial T_{11}}{\partial X_1} + \frac{\partial T_{12}}{\partial X_2} + f_1 = 0$$

$$\frac{\partial T_{12}}{\partial X_1} + \frac{\partial T_{22}}{\partial X_2} + f_2 = 0, \qquad \text{in} \quad \Sigma$$

## POLAR

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) + f_r = 0,$$

$$\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{2}{r}T_{r\theta} + f_\theta = 0, \qquad \text{in} \quad \Sigma$$

$$E_{rr} = \frac{\partial u_r}{\partial r}, \qquad E_{\theta\theta} = \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right),$$

$$E_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right).$$

## CONSTITUTIVE LAW

$$\boxed{\begin{aligned} &\boldsymbol{T}_{\text{int}} = 2\mu \boldsymbol{E}_{\text{int}} + \lambda_* |\boldsymbol{E}_{\text{int}}| \boldsymbol{I}_2 \\ &\lambda_* := \begin{cases} \lambda & \text{(plane strain)} \\ \dfrac{2\lambda\mu}{\lambda + 2\mu} & \text{(plane stress)} \end{cases} \end{aligned}}$$

# THE AIRY STRESS FUNCTION

$$T_{\text{int}} = (\nabla_\perp^2 \Phi) I_2 - \nabla_\perp \otimes \nabla_\perp \Phi$$

satisfies the equilibrium equation identically

$$\nabla_\perp \equiv e_r \frac{\partial}{\partial r} + e_\theta \frac{1}{r}\frac{\partial}{\partial \theta}$$

$$\nabla_\perp \equiv \boldsymbol{e}_1 \frac{\partial}{\partial X_1} + \boldsymbol{e}_2 \frac{\partial}{\partial X_2}$$

CARTESIAN:

$$T_{11} = \frac{\partial^2 \Phi}{\partial X_2^2}, \qquad T_{22} = \frac{\partial^2 \Phi}{\partial X_1^2}, \qquad T_{12} = -\frac{\partial^2 \Phi}{\partial X_1 \partial X_2}$$

$\Phi(X_1, X_2)$

POLAR:

$$T_{rr} = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}, \qquad T_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2}, \qquad T_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Phi}{\partial \theta}\right)$$

$\Phi(r, \theta)$

**Beltrami-Michell** equations give:

$$\boxed{\nabla_\perp^4 \Phi = 0, \qquad \text{in} \quad \Sigma}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}\right) = 0$$

$$\frac{\partial^4 \Phi}{\partial X_1^4} + 2\frac{\partial^4 \Phi}{\partial X_1^2 \partial X_2^2} + \frac{\partial^4 \Phi}{\partial X_2^4} \equiv \left(\frac{\partial^2}{\partial X_1^2} + \frac{\partial^2}{\partial X_2^2}\right)\left(\frac{\partial^2 \Phi}{\partial X_1^2} + \frac{\partial^2 \Phi}{\partial X_2^2}\right) = 0$$

$$\nabla_\perp^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

(2D Laplacian in polar coordinates)

## Relationship between the Cartesian and Polar components (plane elasticity)

$$\boxed{\begin{aligned} T_{rr} &= \frac{1}{2}(T_{11}+T_{22}) + \frac{1}{2}(T_{11}-T_{22})\cos 2\theta + T_{12}\sin 2\theta\,, \\ T_{\theta\theta} &= \frac{1}{2}(T_{11}+T_{22}) - \frac{1}{2}(T_{11}-T_{22})\cos 2\theta - T_{12}\sin 2\theta\,, \\ T_{r\theta} &= \frac{1}{2}(T_{22}-T_{11})\sin 2\theta + T_{12}\cos 2\theta\,. \end{aligned}} \qquad (*)$$

$$\underbrace{\begin{bmatrix} A_{rr} & A_{r\theta} & A_{rz} \\ A_{\theta r} & A_{\theta\theta} & A_{\theta z} \\ A_{zr} & A_{z\theta} & A_{zz} \end{bmatrix}}_{[\boldsymbol{A}]_{POLAR}} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}}_{[\boldsymbol{A}]_{CART}} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

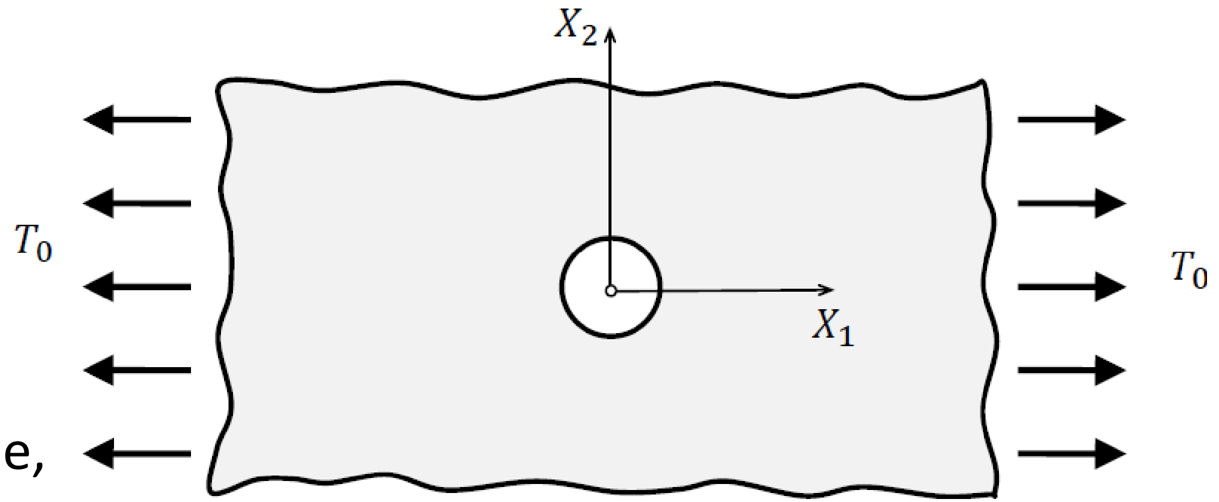
$$[\boldsymbol{A}]_{POLAR} = [\boldsymbol{Q}]^T [\boldsymbol{A}]_{CART} [\boldsymbol{Q}]$$

OBS. If we set $T_{\theta\theta} = T_{r\theta} = 0$ in (*), then

$$T_{11} = T_{rr}\sin^2\theta\,, \qquad T_{22} = T_{rr}\cos^2\theta\,, \qquad T_{12} = T_{rr}\sin\theta\cos\theta\,.$$

## Kirsch's problem

Consider an infinite thin elastic plate with a circular hole of radius $a > 0$, and which is is subjected to uniaxial traction $T_0 > 0$ at infinity, as seen in the sketch on the left.
Assuming that the boundary of the hole is traction-free, it is required to find the stress distribution in the plate.




$$\Sigma \equiv \{(X_1, X_2) \in \mathbb{R}^2 \mid X_1^2 + X_2^2 > a^2\}$$

$$t(-e_r) = 0, \quad \text{on} \quad r = a, \qquad \text{(BC along the rim of the hole)}$$

$$t(\pm e_1) \to \pm T_0 e_1, \quad \text{as} \quad r \to +\infty. \qquad \textbf{(far-field condition)}$$

The far-field condition can be re-cast in the form:

$$T_{rr} \to \frac{T_0}{2}(1 + \cos 2\theta), \quad T_{\theta\theta} \to \frac{T_0}{2}(1 - \cos 2\theta), \quad T_{r\theta} \to -\frac{T_0}{2}\sin 2\theta$$

**No hole:** $$\Phi_0 = \frac{1}{2}T_0 X_2^2 = \frac{1}{4}T_0 r^2(1-\cos 2\theta)$$ easy to check that $\longrightarrow$ $T_{11} = T_0,\ T_{12} = T_{22} = 0$ (i.e., solution)

**With hole:** the **Airy stress function** in this case will be a *perturbation* of the above expression

$$\Phi = \Phi_0(r) + \Phi_2(r)\cos 2\theta$$

$\Phi_0, \Phi_2$ to be found

$$\frac{d^4\Phi_0}{dr^4} + \frac{2}{r}\frac{d^3\Phi_0}{dr^3} - \frac{1}{r^2}\frac{d^2\Phi_0}{dr^2} + \frac{1}{r^3}\frac{d\Phi_0}{dr} = 0$$

$$\frac{d^4\Phi_2}{dr^4} + \frac{2}{r}\frac{d^3\Phi_2}{dr^3} - \frac{9}{r^2}\frac{d^2\Phi_2}{dr^2} + \frac{9}{r^3}\frac{d\Phi_2}{dr} = 0$$

$\Rightarrow$

$$\Phi_0(r) = A_1 r^2 \log r + A_2 r^2 + A_3 \log r + A_4\,,$$

$$\Phi_2(r) = A_5 r^2 + A_6 r^4 + \frac{A_7}{r^2} + A_8\,,$$

**EULER** ODEs

Recall $$T_{rr} = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}\,, \qquad T_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2}\,, \qquad T_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Phi}{\partial \theta}\right)$$

Using the expression of the Airy stress function just found, it follows that

$$T_{rr} = A_1(1+2\log r) + 2A_2 + \frac{A_3}{r^2} - \left(2A_5 + \frac{6A_7}{r^4} + \frac{4A_8}{r^2}\right)\cos 2\theta\,,$$

$$T_{\theta\theta} = A_1(3+2\log r) + 2A_2 - \frac{A_3}{r^2} + \left(2A_5 + 12A_6 r^2 + \frac{6A_7}{r^4}\right)\cos 2\theta\,,$$

$$T_{r\theta} = \left(2A_5 + 6A_6 r^2 - \frac{6A_7}{r^4} - \frac{2A_8}{r^2}\right)\sin 2\theta\,.$$

$T_{rr}, T_{\theta\theta}, T_{r\theta}$ bounded as $r \to \infty$ ⇒ $A_1 = A_6 = 0$ ⇒ simplification

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$$T_{rr} \to 2A_2 - 2A_5 \cos 2\theta\,, \quad T_{\theta\theta} \to 2A_2 + 2A_5 \cos 2\theta\,, \quad T_{r\theta} \to 2A_5 \sin 2\theta \qquad r \to \infty$$

Matching these expressions with the given far-field conditions $\longrightarrow$ $A_2 = -A_5 = T_0/4$

The remaining constants are found by enforcing the BCs on the rim of the hole:

$$\frac{T_0}{2} + \frac{A_3}{a^2} = 0\,, \qquad -\frac{T_0}{2} + \frac{6A_7}{a^4} + \frac{4A_8}{a^2} = 0\,, \qquad -\frac{T_0}{2} - \frac{6A_7}{a^4} - \frac{2A_8}{a^2} = 0 \longrightarrow$$

$$A_3 = -A_8 = -a^2 T_0/2$$

$$A_7 = -a^4 T_0/4$$

$$T_{rr} = \frac{T_0}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{T_0}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta\,,$$

$$T_{\theta\theta} = \frac{T_0}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{T_0}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta\,,$$

$$T_{r\theta} = -\frac{T_0}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta\,.$$

$$T_{\theta\theta}(a,\theta) = T_0(1 - 2\cos 2\theta)$$

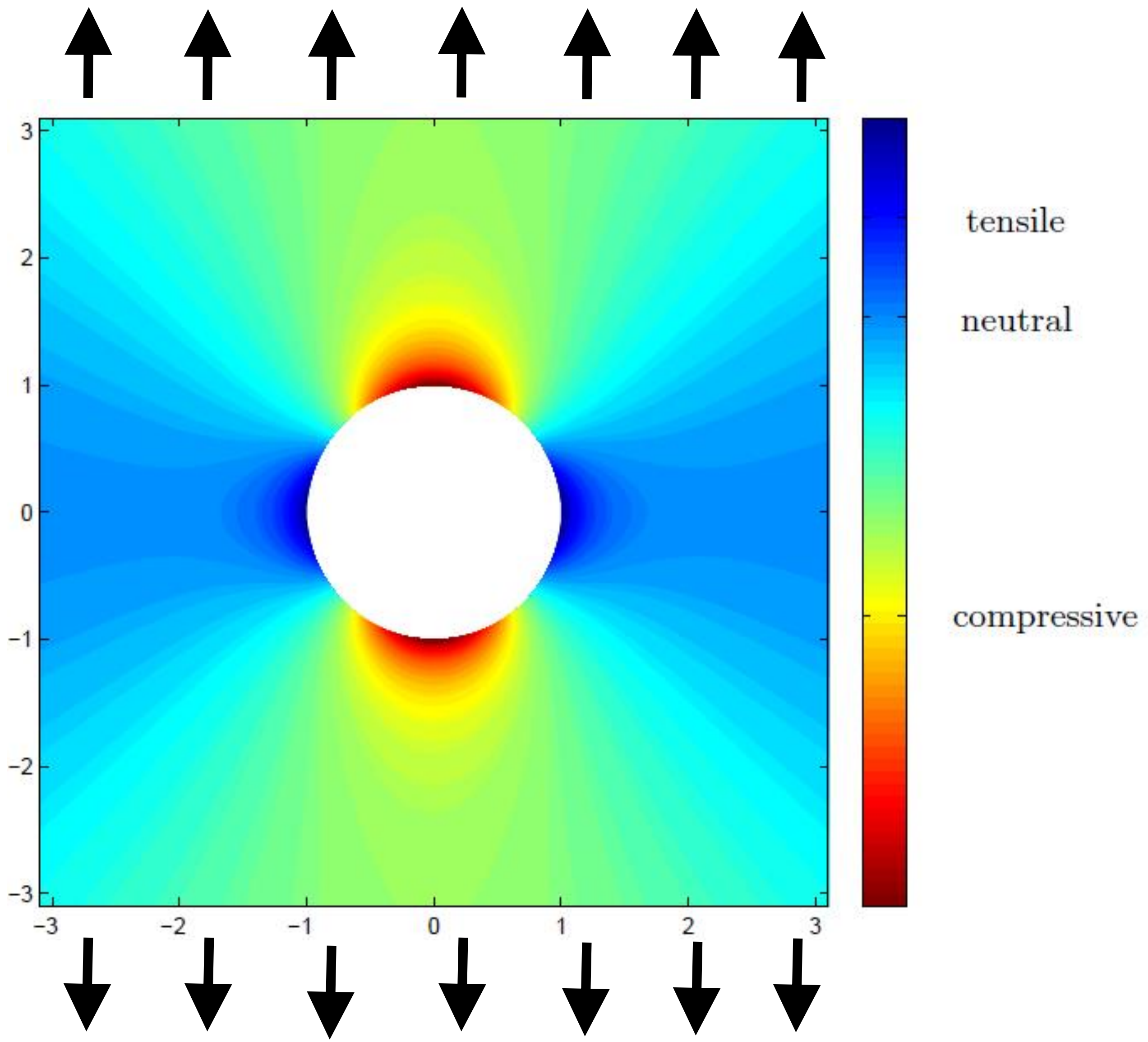
$$T_{\theta\theta}^{MIN} = -T_0$$

$$\theta \in \{0^\circ, 180^\circ\}$$

$$T_{\theta\theta}^{MAX} = 3T_0$$

$$\theta \in \{90^\circ, 270^\circ\}$$

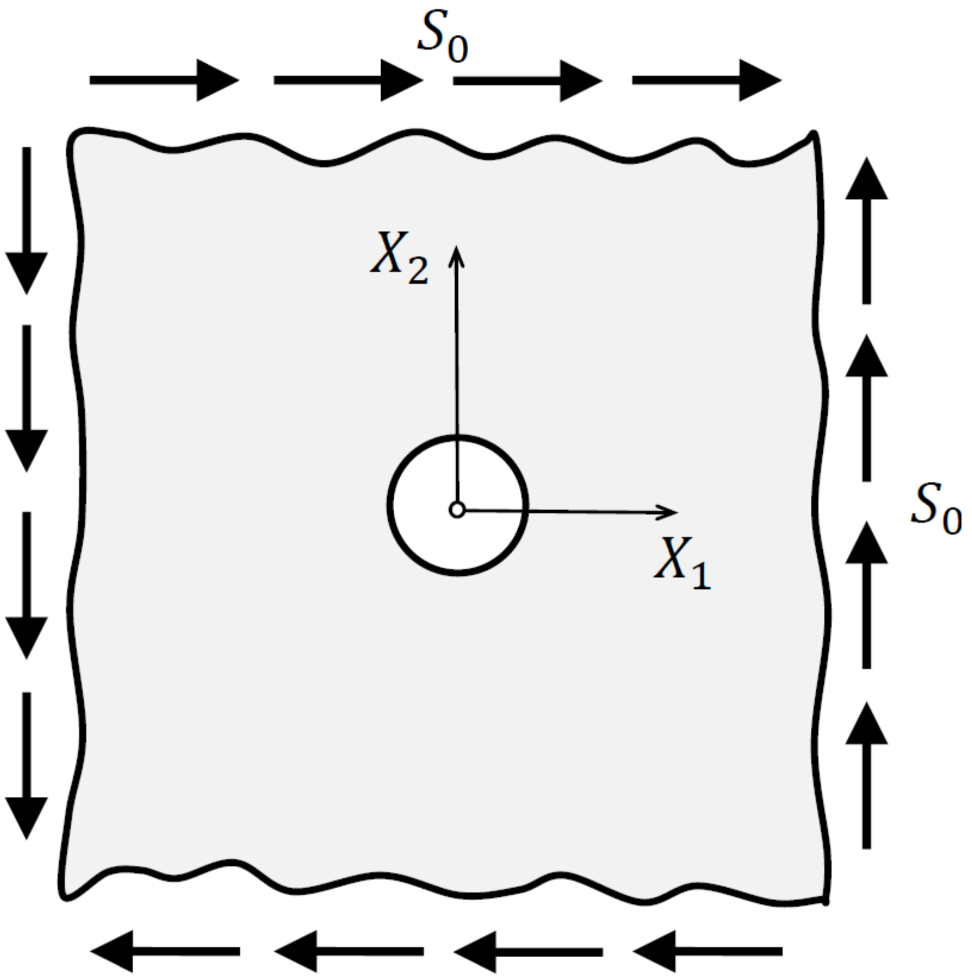
SCF=3


3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3
tensile
neutral
compressive

## Another Kirsch-type problem

Consider an infinite thin elastic plate with a circular hole of radius $a > 0$, and which is is subjected to pure shear at infinity, as seen in the sketch on the left.

Assuming that the boundary of the hole is traction-free, it is required to find the stress distribution in the plate.




**BC**

$$t(-e_r) = 0\,, \qquad \text{on} \quad r = a\,,$$

$$T_{\text{int}} \to S_0(e_1 \otimes e_2 + e_2 \otimes e_1)\,, \qquad \text{as} \quad r \to +\infty$$

The **far-field condition** can be cast in the form:

$$T_{11}, T_{22} \to 0\,, \qquad T_{12} \to S_0\,, \qquad \text{as} \quad r \equiv \left(X_1^2 + X_2^2\right)^{1/2} \to +\infty$$

**No hole:** $\Phi_0 = -S_0 X_1 X_2 = -\frac{1}{2} S_0 r^2 \sin 2\theta$

Far-field condition in polar coordinates: $T_{rr} \to S_0 \sin 2\theta\,, \quad T_{\theta\theta} \to -S_0 \sin 2\theta\,, \quad T_{r\theta} \to S_0 \cos 2\theta$

**With hole:** $\Phi = \Phi_2(r) \sin 2\theta$ → plug in the bi-harmonic equation → Euler-type ODE

$$T_{rr} = -\left(2A_5 + \frac{6A_7}{r^4} + \frac{4A_8}{r^2}\right) \sin 2\theta\,,$$

$$T_{\theta\theta} = \left(2A_5 + 12A_6 r^2 + \frac{6A_7}{r^4}\right) \sin 2\theta\,,$$

$$T_{r\theta} = -\left(2A_5 + 6A_6 r^2 - \frac{6A_7}{r^4} - \frac{2A_8}{r^2}\right) \cos 2\theta$$

⇨

$$T_{rr} = S_0 \left(1 - \frac{4a^2}{r^2} + \frac{3a^4}{r^4}\right) \sin 2\theta\,,$$

$$T_{\theta\theta} = -S_0 \left(1 + \frac{3a^4}{r^4}\right) \sin 2\theta\,,$$

$$T_{r\theta} = S_0 \left(1 + \frac{2a^2}{r^2} - \frac{3a^4}{r^4}\right) \cos 2\theta\,.$$

$T_{\theta\theta}(a, \theta) = -4S_0 \sin 2\theta$ → SCF=4