Introduction to Continuum Mechanics

Your assessment is based on solving any three questions from the list included below.

1. (a) Let $\mathcal{V}$ be the usual three-dimensional Euclidean vector space. For $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d} \in \mathcal{V}$, show that (in the usual notation)

$$(\boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{c} \otimes \boldsymbol{d})^* = (\boldsymbol{a} \wedge \boldsymbol{c}) \otimes (\boldsymbol{b} \wedge \boldsymbol{d})\,; \qquad (1)$$

in particular, if $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are orthonormal, check that

$$\boldsymbol{b} \otimes \boldsymbol{a} = -(\boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{c} \otimes \boldsymbol{c})^*\,.$$

[Hint: you may assume the well-known result: if $\boldsymbol{u}_j$ ($j = 1, 2, 3, 4$) are vectors, then

$$(\boldsymbol{u}_1 \wedge \boldsymbol{u}_2) \cdot (\boldsymbol{u}_3 \wedge \boldsymbol{u}_4) = \begin{vmatrix} \boldsymbol{u}_1 \cdot \boldsymbol{u}_3 & \boldsymbol{u}_1 \cdot \boldsymbol{u}_4 \\ \boldsymbol{u}_2 \cdot \boldsymbol{u}_3 & \boldsymbol{u}_2 \cdot \boldsymbol{u}_4 \end{vmatrix}\,.]$$

(b) Using (1) show that if $\boldsymbol{u} \in \mathcal{V}$ is a unit vector then

$$(\boldsymbol{I} - \boldsymbol{u} \otimes \boldsymbol{u})^* = \boldsymbol{u} \otimes \boldsymbol{u}\,.$$

Assuming now that $\boldsymbol{u} \in \mathcal{V}$ is an arbitrary vector, explain how the right-hand side will have to be modified.

(c) The motion of a deformable body is given, in the usual notation, by $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$. If $\boldsymbol{F}$ is the deformation gradient and J represents the Jacobian of the motion, show from first principles that

$$\frac{\partial}{\partial x_p}(J^{-1}F_{p\alpha}) = 0\,, \qquad (\alpha = 1,\, 2,\, 3)\,.$$

(d) The *Truesdell stress rate* is defined by

$$\mathring{\boldsymbol{\sigma}} \equiv \dot{\boldsymbol{\sigma}} - \boldsymbol{L}\boldsymbol{\sigma} - \boldsymbol{\sigma}\boldsymbol{L}^T + \boldsymbol{\sigma}\,\mathrm{tr}(\boldsymbol{L})\,,$$

where $\boldsymbol{L}$ is the deformation gradient. Show that this stress rate is an objective tensor field.

2. (a) Find all vectors $\boldsymbol{x} \in \mathcal{V}$ such that

$$(\boldsymbol{a} \otimes \boldsymbol{b})(\boldsymbol{a} \otimes \boldsymbol{x}) = (\boldsymbol{x} \otimes \boldsymbol{b})(\boldsymbol{a} \otimes \boldsymbol{a})\,,$$

where $\boldsymbol{a}$, $\boldsymbol{b}$ are arbitrary vectors satisfying $\boldsymbol{a} \cdot \boldsymbol{b} \neq 0$.

(b) For a body held in equilibrium in configuration $\mathcal{B}_c$, show that

$$\int_{\mathcal{B}_c} \boldsymbol{\sigma}\,\mathrm{d}v = \int_{\partial\mathcal{B}_c} (\boldsymbol{\sigma n}) \otimes \boldsymbol{x}\,\mathrm{d}a + \int_{\mathcal{B}_c} \rho(\boldsymbol{b} \otimes \boldsymbol{x})\,\mathrm{d}v\,,$$

where ρ is the mass density of $\mathcal{B}_c$, $\boldsymbol{b}$ represents the body force density, $\boldsymbol{\sigma}$ denotes the Cauchy stress tensor, and $\boldsymbol{n}$ is the outward unit normal on $\partial\mathcal{B}_c$.

(c) If da and dA represent the infinitesimal elements of area in the current and the reference configurations, respectively, show that

$$\frac{da}{dA} = J\sqrt{\boldsymbol{N} \cdot (\boldsymbol{C}^{-1}\boldsymbol{N})}\,,$$

where $\boldsymbol{N}$ is the unit normal associated with dA.

(d) Let $\boldsymbol{x}$ be the position vector of a particle in the current configuration of a deformable body, and consider $\boldsymbol{\sigma}(\boldsymbol{x})$ the Cauchy stress tensor at this point. If

$$\mathcal{U} := \{\boldsymbol{n} \in \mathcal{V},\ |\boldsymbol{n}| = 1\}\,,$$

define the mapping $\phi : \mathcal{U} \longrightarrow \mathbb{R}$ by

$$\phi(\boldsymbol{n}) = |\boldsymbol{\sigma n} - (\boldsymbol{n} \cdot \boldsymbol{\sigma n})\boldsymbol{n}|\,, \qquad (\forall)\,\boldsymbol{n} \in \mathcal{U}\,. \tag{2}$$

Find the extremal values of this function and the vectors for which they are attained.

[Hint: Note that the expression on the RHS of (2) represents the shear stress, so you are asked to identify the directions $\boldsymbol{n}$ for which the shear stress assumes maximum or minimum values. This is an optimization problem with a constraint.]

3. (a) The components of the Cauchy stress tensor in a rectangular Cartesian coordinate system $Ox_1x_2x_3$ at a point P are given in appropriate units by

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 2 \end{bmatrix}\,.$$

i. Find the traction at P on the plane
   - normal to the x_1-axis;
   - whose normal has direction ratios $1 : -3 : 2$;
   - parallel to the plane $x_1 + 2x_2 + 3x_3 = 1$.

ii. Find the principal directions at P and the directions of the principal axes of stress at P. Verify that the principal axes are mutually orthogonal.

iii. Consider now a change of coordinates according to the transformation

$$\overline{x}_1 = \frac{1}{3}(x_1 - 2x_2 + 2x_3)\,,$$

$$\overline{x}_2 = \frac{1}{3}(-2x_1 + x_2 + 2x_3)\,,$$

$$\overline{x}_3 = \frac{1}{3}(-2x_1 - 2x_2 - x_3)\,.$$

Verify that this defines an orthogonal transformation, and find the components of the stress tensor defined above in the new coordinate system. Use this answer to check the results obtained at (ii) above.

(b) For the homogeneous deformation $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$ defined by

$$x_1 = \alpha X_1 + \beta X_2\,, \qquad x_2 = -\alpha X_1 + \beta X_2\,, \qquad x_3 = \mu X_3\,, \tag{3}$$

where α, β, and μ are positive constants, determine the component representation of the right Cauchy-Green tensor and the principal stretches. Find $\boldsymbol{R}$ and $\boldsymbol{U}$ for the polar decomposition of the deformation gradient associated with (3).

(c) Consider the following stress distribution for a circular cylindrical bar of length $L > 0$, having the 1-direction as its axis,

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 0 & -\alpha x_3 & \alpha x_2 \\ -\alpha x_3 & 0 & 0 \\ \alpha x_2 & 0 & 0 \end{bmatrix}.$$

i. If the lateral surface of the cylinder is given by the equation $x_2^2 + x_3^2 = 4$, find the distribution of the traction vector on this surface.

ii. Find also the expression of the traction vector on the ends $x_1 = 0$ and $x_1 = L$, and then calculate the corresponding resultant forces and moments. Discuss your results.

4. (a) An elliptical bar of finite length has a lateral surface defined by the equation $x_2^2 + 2x_3^2 = 1$ and has the following stress distribution,

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 0 & -2x_3 & x_2 \\ -2x_3 & 0 & 0 \\ x_2 & 0 & 0 \end{bmatrix}.$$

Show that the lateral surface of the cylinder is traction-free, and then find the resultant force and resultant moment about the origin O of the stress vector on the left end face $x_1 = 0$.

(b) A sphere of unit radius and centred at the origin of coordinates is subjected to the deformation $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X})$, with

$$\boldsymbol{\chi}(\boldsymbol{X}) := (X_1 + \varepsilon X_2 X_3)\boldsymbol{e}_1 + (X_2 + \varepsilon X_3 X_1)\boldsymbol{e}_2 + (X_3 + \varepsilon X_1 X_2)\boldsymbol{e}_3\,,$$

where $\varepsilon > 0$ is a constant parameter. Show that the volume of the sphere after deformation is

$$\frac{4\pi}{3}\left(1 - \frac{3}{5}\varepsilon^2\right).$$

(c) Let $\boldsymbol{W}$ be a skew-symmetric tensor associated with the unit axial vector $\boldsymbol{w}$, and let $\alpha \in \mathbb{R}$. Show that

$$\det(\boldsymbol{I} + \alpha \boldsymbol{W}) = 1 + \alpha^2\,,$$

and that

$$(\boldsymbol{I} + \alpha \boldsymbol{W})^{-1} = (1 + \alpha^2)^{-1}(\boldsymbol{I} + \alpha \boldsymbol{A} + \alpha^2 \boldsymbol{B})\,,$$

where $\boldsymbol{A}$, $\boldsymbol{B}$ are second-order tensors that you must specify.

[Hint: For the second part use the Cayley-Hamilton Theorem.]

5. (a) If the motion of a deformable body is given (in standard notation) by $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X},\, t)$, with

$$\begin{cases} x_1 = X_1 + X_3\,t^3\,, \\ x_2 = X_2 + X_1\,t^3\,, \\ x_3 = X_3 + X_2\,t^3\,, \end{cases}$$

then find the velocity of

- the particle which was at the point $\boldsymbol{X}^0 = (4,\, 4,\, 4)$ at the reference time $t = 0$.
- the particle which occupies the point $\boldsymbol{x}^0 = (4,\, 4,\, 4)$ at time $t > 0$.

(b) Determine the eigenvalues, the eigenvectors, and a spectral representation for the following tensors

$$\boldsymbol{A} = \alpha \boldsymbol{I} + \beta \boldsymbol{m} \otimes \boldsymbol{m}\,,$$

$$\boldsymbol{B} = \boldsymbol{m} \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \boldsymbol{m}\,.$$

Here α and β are scalars, while $\boldsymbol{m}$ and $\boldsymbol{n}$ are orthogonal unit vectors.

(c) Assuming that the body force $\mathbf{b}$ is objective, show that Cauchy's equation of motion

$$\mathrm{div}\boldsymbol{\sigma} + \rho \boldsymbol{b} = \rho \boldsymbol{a}$$

is not invariant under all changes in observer. Find non-trivial sufficient conditions on the change of observer for which the invariance does hold.

(d) It was shown in the solved examples for Chapter 1 that the field equations describing the deformation of a linearly isotropic solid can be formulated in terms on the displacement field $\boldsymbol{u} \equiv \boldsymbol{u}(\boldsymbol{x})$, namely,

$$\mu \nabla^2 \boldsymbol{u} + (\lambda + \mu)\mathrm{grad}(\mathrm{div}\boldsymbol{u}) + \boldsymbol{f} = \boldsymbol{0}\,,$$

where $\boldsymbol{f} := \rho \boldsymbol{b}$ is the body force per unit volume. Show that this equation can be also cast in the form

$$\nabla^2 \boldsymbol{u} + \frac{1}{1 - 2\nu}\mathrm{grad}\,(\mathrm{div}\boldsymbol{u}) + \frac{1}{\mu}\boldsymbol{f} = \boldsymbol{0}\,, \qquad (4)$$

where ν is the usual Poisson's ratio, and then check that the vector field

$$\boldsymbol{u} = \boldsymbol{\Psi} - \frac{1}{4(1-\nu)}\mathrm{grad}\,(\boldsymbol{x} \cdot \boldsymbol{\Psi} + \phi)$$

is a solution of (4), provided that

$$\nabla^2 \boldsymbol{\Psi} = -\frac{1}{\mu}\boldsymbol{f}\,, \qquad \nabla^2 \phi = \frac{1}{\mu}\boldsymbol{x} \cdot \boldsymbol{f}\,,$$

where Ψ and ϕ are assumed to be infinitely differentiable.

DEADLINE: *Please follow the instructions on the MAGIC website.*